Justifications that Might Be Wrong

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In an unpublished paper, “A logic for evidence and truth”, Walter Carnielli and Abilio Rodrigues introduce a natural deduction system in which it is not truth, but evidence that is preserved.
Their system is called the *Basic Logic of Evidence*.

Carnielli and Rodrigues build another system on top of this, but we will not consider it here.

From now on, *BLE* stands for Basic Logic of Evidence.
In their treatment, evidence might be incomplete or contradictory.

Here’s a (very) simple example of a rule.
\[
\begin{array}{c}
A \\ \hline \\
B \\
\hline \\
A \land B
\end{array}
\]

Don’t think of this as saying:
If \(A\) and \(B\) are true, so is \(A \land B\).

Instead, quoting from the paper:

if \(\kappa\) and \(\kappa'\) are evidence, respectively, for \(A\) and \(B\),
\(\kappa\) and \(\kappa'\) together constitute evidence for \(A \land B\).
There is hidden machinery here.

“$\kappa$ and $\kappa'$ together constitute evidence”

Somehow, evidence can be combined.

How?

What are we talking about?

I have a specific proposal to show you.
My central tool is a justification logic.

The first justification logic was LP, introduced by Artemov.

Gödel showed intuitionistic logic embedded into S4.
Artemov created justification logic LP with formal proof terms.

He showed S4 embedded into LP, and LP embedded into Peano arithmetic.
However:

LP contains *proof terms*. These are abstractions of formal proofs. It is the wrong thing for us. Carnielli and Rodrigues want reasons that can be incomplete or incorrect. Like everyday reasons.
Our plan.

1. Create a modal logic in which the modal operator captures “has evidence” in the way that S4 does for “has a formal proof”. Think of this as representing implicit evidence.
2. Show the Carnielli and Rodrigues system embeds into that modal logic.
3. Create a justification logic, like LP, but in which justification terms represent evidence that may not be factive. Think of this as representing explicit evidence.
4. Show the modal logic embeds into the justification logic.
Then we can formally understand the Carnielli and Rodrigues notion of evidence by its representation in our justification logic.
But first, what is the Carnielli and Rodrigues system \textit{BLE}?
BLE
(Basic Logic of Evidence)

Rules for $\land$, $\lor$ and $\supset$
 turn out to be intuitionistic.

Intuitionistic negation,
or $\bot$, is not appropriate.
The notion of evidence usually considered appropriate for intuitionistic logic is *proof*.

Evidence that it is not raining could be:
I looked out the window and saw it was not raining.

*Direct* evidence for a negated statement.

This kind of evidence is weaker than formal proof, and can be contradictory or incomplete.
Instead of intuitionistic negation, \textit{BLE} has introduction and elimination rules for Nelson’s strong negation.

\textit{BLE} does \textit{not} have
\begin{align*}
X \lor \neg X \text{ or } \neg(X \land \neg X).
\end{align*}

In fact, \textit{BLE} is equivalent to the paraconsistent Nelson logic \textit{N4}.
Then, here is BLE axiomatically.

\[
\begin{align*}
A1 & \quad P \supset (Q \supset P) \\
A2 & \quad (P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R)) \\
A3 & \quad (P \land Q) \supset P \\
A4 & \quad (P \land Q) \supset Q \\
A5 & \quad (P \supset Q) \supset ((P \supset R) \supset (P \supset (Q \land R))) \\
A6 & \quad P \supset (P \lor Q) \\
A7 & \quad Q \supset (P \lor Q) \\
A8 & \quad (P \supset R) \supset ((Q \supset R) \supset ((P \lor Q) \supset R)) \\
A9 & \quad \neg
\neg P \equiv P \\
A10 & \quad \neg(P \lor Q) \equiv (\neg P \land \neg Q) \\
A11 & \quad \neg(P \land Q) \equiv (\neg P \lor \neg Q) \\
A12 & \quad \neg(P \supset Q) \equiv (P \land \neg Q)
\end{align*}
\]
I’ll continue calling this \textit{BLE}, and not \textit{N4}.

There are already semantics for it.

What we want is a \textit{formal evidence based} interpretation for it, where evidence might be wrong.
You can’t go wrong by following Gödel.

He embedded Intuitionistic Logic into modal S4.

He thought of □ as provability.
A proof is the strongest kind of evidence.
Gödel’s Translation
(one version)

\[ P^f = \Box P \ (P \text{ atomic}) \]

\[ (X \land Y)^f = \Box (X^f \land Y^f) \]

\[ (X \lor Y)^f = \Box (X^f \lor Y^f) \]

\[ (\sim X)^f = \Box \neg X^f \]

\[ (X \supset Y)^f = \Box (X^f \supset Y^f) \]

\sim \text{ is intuitionistic negation}
\f \text{ indicates evidence for}
As everybody knows, $X$ is intuitionistically provable iff $X^f$ is S4 provable.

But we have two problems.

$\sim$ is the wrong negation.

We’ll throw it away shortly.

S4 is the wrong modal logic.

This is of more ideological significance.
S4 can’t handle faulty evidence. 
$\Box X \supset X$ is an axiom.

Proofs are factive, evidence generally need not be.

We replace this with 
$\Box \Box X \supset \Box X$
(often called axiom scheme $X$).

Evidence that there is evidence for something counts as evidence.
Our logic of (implicit) evidence is KX4:

**Tautologies**

\[ \Box (X \supset Y) \supset (\Box X \supset \Box Y) \]
\[ \Box \Box X \supset \Box X \]
\[ \Box X \supset \Box \Box X \]

**Modus Ponens**

\[ \frac{X \quad X \supset Y}{Y} \]

**Necessitation**

\[ \frac{X}{\Box X} \]
Semantics for $\Box X \supset \Box X$

is a \textit{density} condition.

And for $\Box X \supset \Box \Box X$

is a \textit{transitivity} condition.

Except for the next slide,
we make no use of semantics.
A KX4 Model

\[ [0, +\infty) \]

possible worlds, left closed interval.

accessibility relation 

\(<\)

We have transitivity and denseness. This is a KX4 frame.

(We even have seriality.)
Let $P$ be true at all worlds except 0.

Then $\Box P$ is true at 0, but $P$ is not.

So $\Box P \supset P$ is not a theorem of KX4.

$\text{KX4} \neq \text{S4}$
$$□□X \supset □X$$ is an instance of $$□Y \supset Y$$

So every theorem of KX4 is a theorem of S4.

We have KX4 ⊆ S4.
In KX4 we have $\Box\Box X \equiv \Box X$.

KX4 is a normal modal logic, so we have substitutivity of equivalences.

So, replacing any positive number of consecutive $\Box$ symbols by a single one turns a theorem of KX4 into a theorem.
\(\Box(\Box \Box P \supset \Box(\Box Q \supset \Box P))\) is a theorem of K.

Hence of KX4.

Hence so is
\(\Box(\Box P \supset \Box(\Box Q \supset \Box P))\).

This is \((P \supset (Q \supset P))^f\).
The following are theorems of K.

• \( \Box \Box (\Box P \supset \Box (\Box Q \supset \Box R)) \supset \Box (\Box (\Box P \supset \Box \Box Q) \supset \Box (\Box P \supset \Box \Box R)) \)

• \( \Box \Box \Box P \supset \Box (\Box \Box Q \supset \Box (\Box P \land \Box Q)) \)

• \( (\Box P \land \Box Q) \supset \Box P \)

• \( (\Box P \land \Box Q) \supset \Box Q \)

• \( \Box \Box P \supset \Box (\Box P \lor \Box Q) \)

• \( \Box \Box Q \supset \Box (\Box P \lor \Box Q) \)

• \( \Box \Box \Box (\Box P \supset \Box R) \supset \Box (\Box \Box (\Box Q \supset \Box R) \supset \Box (\Box (\Box P \lor \Box Q) \supset \Box \Box R)) \)
It follows that in KX4 we have:

- \(((P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R)))^f\)
- \((P \supset (Q \supset (P \land Q)))^f\)
- \(((P \land Q) \supset P)^f\)
- \(((P \land Q) \supset Q)^f\)
- \((P \supset (P \lor Q))^f\)
- \((Q \supset (P \lor Q))^f\)
- \(((P \supset R) \supset ((Q \supset R) \supset ((P \lor Q) \supset R)))^f\)
And Modus Ponens?

Note that $Z^f$ has an initial $\Box$, so $Z^f \equiv \Box Z^f$.

$X^f$
$\Box X^f$
$(X \supset Y)^f$
$\Box (X^f \supset Y^f)$
$\Box X^f \supset \Box Y^f$
$\Box Y^f$
$Y^f$
Then if $X$ is an intuitionistic theorem without negation, $X^f$ is a theorem of KX4.

Also if $X^f$ is a theorem of KX4 it is a theorem of S4, hence (well-known result) it is a theorem of Intuitionistic logic.

The Gödel translation embeds Intuitionistic logic without negation into KX4.
Side Remarks on Intuitionistic Negation
We won’t want intuitionistic negation. But here are some comments anyway.

Intuitionistically, \( \sim X \equiv (X \supset \bot) \).

\[
(\sim P)^f = \Box \neg \Box P \\
= \Box (\Box P \supset \bot)
\]

\[
(P \supset \bot)^f = \Box (\Box P \supset \Box \bot)
\]

In S4 \( \bot \) and \( \Box \bot \) are equivalent. In KX4 they are not.
If we define $\sim X$ to mean $X \supset \bot$ and use the Gödel translation we have

- $\Box \bot \supset \Box P$
- $\Box \Box \Box (\Box P \supset \Box \Box Q) \supset$
  $\Box (\Box (\Box P \supset \Box (\Box Q \supset \Box \bot)) \supset \Box (\Box P \supset \Box \Box \bot))$
- $\Box \Box \Box P \supset \Box (\Box (\Box P \supset \Box \bot) \supset \Box \Box Q)$

in $K$,
And so, in KX4:

- \((\bot \supset P)^f\)
- \(((P \supset Q) \supset ((P \supset \neg Q) \supset \neg P))^f\)
- \((P \supset (\neg P \supset Q))^f\)

So we could handle intuitionistic negation. But what we want is *strong* negation.

From now on, no intuitionistic negation.
This ends the side remarks.

Side Remarks on
Intuitionistic Negation
Step 3

Justification Logics

Justification logics are modal-like logics in which necessity is replaced with explicit justification terms.
The first justification logic was LP, due to Artemov. It corresponded to modal S4.

Instead of $\square X$, in LP we have $t:X$

Not: $X$ is provable, (Gödel’s reading), but $X$ is so with justification (proof) $t$. 
LP, like S4, has an axiomatic presentation. It also has a possible world semantics, but we’ll ignore that today.

S4 and LP correspond. Here’s what that means.
The Forgetful Functor

\[ X \rightarrow X^\circ \]

LP formula \quad Modal formula

Replace all justification terms with \( \Box \) occurrences.

Easy part: If \( X \) is a theorem of LP then \( X^\circ \) is a theorem of S4.

True for axioms, preserved by \textit{modus ponens}.
A normal realization of a modal formula replaces each occurrence of $\Box$ by a justification term. Negative occurrences of $\Box$ are replaced with distinct justification variables; positive occurrences are replaced by justification terms built from them.

Note: if $Y$ is a normal realization of $X$, then $Y^\circ = X$. 
Hard part: Every theorem of S4 has a normal realization that is a theorem of LP.

The forgetful functor maps theorems of LP onto theorems of S4.

Realizations provide flow of information details for S4 theorems.

Examples later.
Since Artemov’s work, many modal logics have been shown to have justification counterparts.

The family is infinite, in fact.

We need one for KX4.

Now we get to details.
Step 4
Justification Counterpart for KX4

First, justification terms are built up as follows:
• *justification variables*, \( x, y, \ldots \).
  Every justification variable is a justification term.

• *justification constants*, \( a, b, \ldots \).
  Every justification constant is a justification term.

• *binary operation symbols*, \( + \) and \( \cdot \).
  If \( u \) and \( v \) are justification terms, so are \((u + v)\) and \((u \cdot v)\).

• *unary operation symbol*, \( ! \).
  If \( t \) is a justification term, so is \(!t\).

• *binary operation symbol*, \( c \).
  If \( t \) and \( u \) are justification terms, so is \((t \circ u)\).
Variables stand for arbitrary justification terms, and can be substituted for under certain circumstances.

Constants stand for reasons that are not further analyzed typically axioms.

· corresponds to *modus ponens*.

If $X \supset Y$ is so for reason $s$ and $X$ is so for reason $t$, then $Y$ is so for reason $s \cdot t$. 


+ is a kind of weakening. If \( X \) is so for either reason \( s \) or reason \( t \), then \( s + t \) is also a reason for \( X \).

\( ! \) operation is a \textit{justification checker}. If we have \( t:X \) then \( !t \) verifies this, so we have \( !t:t:X \).

\( c \) is a \textit{combination operation}. A justification for a justification for \( X \) can be combined into a simple justification for \( X \). If we have \( \triangledown u:X \) then we also have \( [t \cdot c \cdot u]:X \).
*Justification formulas* are built up from propositional letters using propositional connectives together with the formation rule:

if \( t \) is a justification term and \( X \) is a justification formula, then \( t:X \) is a justification formula.
Axioms

- All tautologies (or enough of them)
- \( s:(X \supset Y) \supset (t:X \supset [s \cdot t]:Y) \)
- \( s:X \supset [s + t]:X \) and \( t:X \supset [s + t]:X \)
- \( s:X \supset !s:s:X \)
- \( s:t:X \supset [s \cdot t]:X \)

Rules

- \( X, X \supset Y \Rightarrow Y \)

We call this logic JX4.
Constant Specifications

Let $C$ be a set of formulas of the form $c:X$ where $c$ is a constant and $X$ is an axiom.

$C$ is a constant specification.

We assume about $C$ that each axiom has a constant justifying it (axiomatically appropriate) and all instances of the same axiom scheme have the same constant (schematic).
\( \vdash_{JX4(C)} X \) means there is a sequence of JX4 formulas in which each is either an axiom of JX4, a member of \( \mathcal{C} \), or follows from earlier formulas by modus ponens.
Internalization

The following has a constructive proof and takes the place of necessitation.

If \( \vdash_{JX4(C)} X \) then for some justification term \( t \),
\[
\vdash_{JX4(C)} t : X.
\]

\( t \) can be built from constants using only \( \cdot \) and \( ! \).
It is easy to see that for a justification formula $X$, $\vdash_{JX4(C)} X$ implies $\vdash_{KX4} X^\circ$.

It can be checked for axioms and it is preserved by modus ponens.

There is also a converse.
The axioms of KX4 are Geach formulas, of the form $\Diamond^{k\Box l}X \supset \Box^{m\Diamond n}X$

I proved a very general result in *Justification logics and realization*, Technical Report TR-2014004, CUNY Ph.D. Program in Computer Science, March 2014.

In part it says that modal logics axiomatized by Geach formulas always have realization theorems.

In particular it tells us

If $\vdash_{KX4} Y$ then there is a justification formula $X$ such that $\vdash_{JX4(C)} X$ and $Y = X^\circ$.

Every theorem of $KX4$ has a provable normal realization in $JX4(C)$. 
Example

Here is a provable formula of KX4.
(It has a simpler proof in T, but we don’t have factivity in KX4.)

\[ \Box(\Box P \supset \Box Q) \supset (\Box P \supset \Box Q) \]

Here is a JX4 realization.

\[ v_1:(v_2:P \supset v_3:Q) \supset (v_2:P \supset [(v_1 \cdot !v_2) c v_3]:Q) \]
Propositional intuitionistic logic embeds into KX4. Now KX4 embeds into JX4. So, we can understand propositional intuitionistic logic not only as a logic of proof, but as a logic of evidence, which may be non-factual, uncertain, or contradictory.
Now let’s drop intuitionistic negation, and add strong negation, to get BLE, and let’s treat this in a rather well-known way, but in our present context.
Step 5
Adding Strong Negation

From intuitionistic logic, drop negation \( \sim \), getting positive logic. Add strong (boolean) negation \( \neg \), getting Nelson N4, here called BLE, basic logic of evidence.
Think of $\neg X$ as so if there is ‘direct’ evidence that $X$ is not so.

We observe that it is not raining.
We feel that we are not happy.

Observational evidence can be wrong, or missing.
Now our embedding of BLE into KX4 is extended to include strong negation.

This can be done indirectly or directly.

Each has its uses.

First, indirectly. There is a negation normal form. It drives negations to the atomic level, using the following boolean equivalences.
$(X \land Y)^N = X^N \land Y^N$

$(\neg(X \land Y))^N = (\neg X)^N \lor (\neg Y)^N$

$(X \lor Y)^N = X^N \lor Y^N$

$(\neg(X \lor Y))^N = (\neg X)^N \land (\neg Y)^N$

$(X \supset Y)^N = X^N \supset Y^N$

$(\neg(X \supset Y))^N = X^N \land (\neg Y)^N$

$(\neg\neg X)^N = X^N$
Then to interpret a BLE formula $X$ containing strong negations, we work with $X^N$.

But we still need to model negated atoms.
For this a standard device is used.

For each propositional letter $P$
we introduce a new ‘dual’ letter $\overline{P}$,
meant to represent the opposite of $P$.

The two are treated independently
in the semantics.
There is a four-valued structure here. At a world we could have:
both $P$ and $\overline{P}$
$P$ but not $\overline{P}$
not $P$ but $\overline{P}$
neither $P$ nor $\overline{P}$. 
So, continue negation normal form translation by adding the following.

\[ P^N = P \]

\[ (\neg P)^N = \overline{P} \]

Thus strong negation is translated away but the language is extended with new atoms.

With negations gone, our earlier embedding work applies.
Equivalently, there is a handy direct approach.

In addition to the positive embedding from before, $X$ embeds as $X^f$  
(evidence for)

We have a negative embedding  
$X$ embeds as $X^a$  
(evidence against)
\[(X \land Y)^a = \Box(X^a \lor Y^a)\]  
\[(X \lor Y)^a = \Box(X^a \land Y^a)\]  
\[(X \supset Y)^a = \Box(X_f^a \land Y^a)\]  
\[\neg X)^a = X_f^a\]  
\[\neg X)^f = X^a\]  
\[P^a = \Box \overline{P}\]

One can show by induction on degree
\[X_f^a = (X^N_f)^f\]
Here’s a translation example, just to show how it works.

\[(P \supset \neg(Q \supset \neg(R \supset \neg S)))^f\]
\[= \square(P^f \supset (\neg(Q^f \supset \neg(R^f \supset \neg S))^f))\]
\[= \square(\square P \supset (Q^f \supset \neg(R^f \supset \neg S))^a)\]
\[= \square(\square P \supset (\square(\square Q \supset (R^f \supset \neg S)^f)))\]
\[= \square(\square P \supset (\square(\square Q \supset (\square R \supset S^a))))\]
\[= \square(\square P \supset (\square(\square Q \supset (\square R \supset \square \overline{S}))))\]
So, here’s the overall structure.

1. Begin with a formula of BLE, allowing strong negations.
2. Translate this into a modal formula, eliminating negations, but adding more propositional atoms.
3. Interpret this formula in KX4, to get an implicit evidence interpretation.
4. Realize this into JX4, to get an explicit evidence interpretation.
An Example

\[ \neg (P \supset Q) \supset (Q \supset P) \]

is provable in BLE.

We give an implicit and an explicit evidence interpretation for it.
Apply the ‘evidence for’ translation.

\[
\neg(P \supset Q) \supset (Q \supset P)^f
\]

\[
= \Box[(\neg(P \supset Q))^f \supset (Q \supset P)^f]
\]

\[
= \Box[(P \supset Q)^{\alpha} \supset (Q \supset P)^f]
\]

\[
= \Box[\Box(P^f \land Q^\alpha) \supset \Box(Q^f \supset P^f)]
\]

\[
= \Box[\Box(P \land \Box \overline{Q}) \supset \Box(\Box Q \supset \Box P)]
\]
Then
\[\Box[(\Box P \land \Box \neg Q) \supset \Box(\Box Q \supset \Box P)]\]

is our *implicit* evidence reading of
\[\neg(P \supset Q) \supset (Q \supset P)\]
We have evidence for $P$ and evidence against $Q$

Evidence for $Q$ entails that there is evidence for $P$
\[ \Box (\Box P \land \Box \overline{Q}) \quad \Box (\Box Q \supset \Box P) \]

We have evidence for $P$ and evidence against $Q$

Evidence for $Q$ entails that there is evidence for $P$

There is evidence for this situation

There is evidence for this situation
We have evidence the first situation entails the second.

\[ \Box(\Box P \land \Box \overline{Q}) \supset \Box(\Box Q \supset \Box P) \]

We have evidence for \( P \) and evidence against \( Q \)

Evidence for \( Q \) entails that there is evidence for \( P \)

There is evidence for this situation

There is evidence for this situation
$$\Box[\Box(\Box P \land \Box \lnot Q) \supset \Box(\Box Q \supset \Box P)]$$

None of this tells us anything about what kind of evidence we may have.

But this is provable in KX4.
Now realize
\[
\Box[\Box(\Box P \land \Box \overline{Q}) \supset \Box(\Box Q \supset \Box P)]
\]
in JX4.

We get
\[
t_2:[v_4:(v_1:P \land v_2:Q) \supset (t_1 \cdot v_4):(v_3:Q \supset v_1:P)]
\]
where \(t_1\) and \(t_2\) are built from constants using \(\cdot\) and \(!\).
Positive locations

\[ t_2: [v_4: (v_1: P \land v_2: \overline{Q}) \supset (t_1 \cdot v_4): (v_3: Q \supset v_1: P)] \]

Negative locations
$t_1$ and $t_2$ are from the Internalization Theorem.

$$(v_1:P \land v_2:\overline{Q}) \supset (v_3:Q \supset v_1:P)$$

is provable in JX4.

$t_1$ is a justification for it.
\[ v_4 : (v_1 : P \land v_2 : \overline{Q}) \supset (t_1 \cdot v_4) : (v_3 : Q \supset v_1 : P) \]

is provable in JX4.

\[ t_2 \] justifies it.

We don’t need the details.
Note that in
\[ t_2: [v_4:(v_1:P \land v_2:\overline{Q}) \supset (t_1 \cdot v_4):(v_3:Q \supset v_1:P)] \]
\(v_2\) and \(v_3\) appear only once.

After introduction, they are never used.
They don’t matter.

Explicit evidence can give us information that is hidden when implicit evidence is used.
Another Example

\[\neg(P \land \neg P) \supset ((\neg P \supset P) \supset P]\]

is provable in BLE.

Embedding into modal language we get
\[\Box[\Box(\Box\neg P \lor \Box P) \supset \Box(\Box(\Box \neg P \supset \Box P) \supset \Box P)]\]

and this formula involving implicit evidence

is provable in KX4.
A realization of this, provable in JX4, is the following.

\[ t_4: \{ v_5: (v_2: P \lor v_1: \overline{P}) \supset [t_3 \cdot v_5]: (v_4: (v_2: \overline{P} \supset v_3: P) \supset (((t_2 \cdot !v_2 \cdot v_4) \cdot c \cdot v_3) + (t_1 \cdot !v_1 \cdot v_4) \cdot c \cdot v_1): P) \} \]
\[ t_1 \text{ justifies } v_1:P \supset ((v_2:\overline{P} \supset v_3:P) \supset v_1:P) \]

\[ t_2 \text{ justifies } v_2:\overline{P} \supset ((v_2:\overline{P} \supset v_3:P) \supset v_3:P) \]

\[ t_3 \text{ justifies } (v_2:\overline{P} \vee v_1:P) \supset (v_4:(v_2:\overline{P} \supset v_3:P) \supset [((t_2 \cdot !v_2 \cdot v_4) \cdot c \cdot v_3) + (t_1 \cdot !v_1 \cdot v_4) \cdot c \cdot v_1])]:P \]

\[ t_4:\{v_5:(v_2:P \vee v_1:\overline{P}) \supset [t_3 \cdot v_5]:v_4:(v_2:\overline{P} \supset v_3:P) \supset [((t_2 \cdot !v_2 \cdot v_4) \cdot c \cdot v_3) + (t_1 \cdot !v_1 \cdot v_4) \cdot c \cdot v_1])]:P \} \]
Negative Positions

\[
t_4: \{ v_5: (v_2: P \vee v_1: \overline{P}) \supset [t_3 \cdot v_5]: (v_4: (v_2: \overline{P} \supset v_3: P) \supset [(t_2 \cdot !v_2 \cdot v_4) \circ v_3) + (t_1 \cdot !v_1 \cdot v_4) \circ v_1]) : P) \}
\]

These involve variables

\[ v_1, v_2, v_3, v_4, v_5 \]
$v_5$ is used here

\[
t_4:\{v_5:(v_2:P \lor v_1:\overline{P}) \supset [t_3 \cdot v_5]:(v_4:(v_2:\overline{P} \supset v_3:P) \supset \big( ((t_2 \cdot !v_2 \cdot v_4) \circ v_3) + (t_1 \cdot !v_1 \cdot v_4) \circ v_1 \big)]:P) \}
\]

$v_1, v_2, v_3, v_4$ are used here
Each input variable plays a role in this example.

\[ t_4: \{ v_5: (v_2: P \lor v_1: \overline{P}) \supset [t_3 \cdot v_5]: (v_4: (v_2: \overline{P} \supset v_3: P) \supset [(t_2 \cdot v_2 \cdot v_4) \cdot c v_3) + (t_1 \cdot !v_1 \cdot v_4) \cdot c v_1]) : P \} \]
What Is Still Needed?

Two main items are still missing.
(Future research)
First, Carnielli and Rodrigues build a second logic on top of BLE. In it there is an operator that, roughly speaking, identifies a subformula as classical.

\[\neg(P \land \neg P) \supset ((\neg P \supset P) \supset P)\]

\(P\) behaves classically
But this is not exactly what they have in mind.

Something more is involved.

Work is needed here.
Second, there is a Realization Theorem connecting $KX4$ and $JX4$. But, it has a non-constructive proof. Sometimes we can get a constructive proof, such as the Realization Theorem connecting $S4$ and $LP$. 
Constructive Realization proofs make use of cut free proof systems, but it is not known yet whether cut free proof systems for KX4 (there are some) will work for this.

Work is needed here too.
But that’s enough for now.

Thank You