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ELEMENTARY FORMAL SYSTEMS FOR HYPERARITHMETICAL RELATIONS

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§ 1. Introduction

Elementary formal systems, as defined in [4], provide an elegant approach to recursion theory. In this paper we show how, by a simple modification, they may likewise serve for hyperarithmetic theory. Out of this work we derive a curious model-theoretic characterization of both the Π_1^1 sets and the r.e. sets.

As defined in [4], elementary formal systems deal with concatenation of words over a finite alphabet. Actually, it is possible to construct elementary formal systems to deal directly with any mathematical structure. We find it convenient here, though not necessary, to work with elementary formal systems that act on numbers rather than words. In § 2 we define the system we will be using, but we observe now that what we do can easily be modified to apply to elementary formal systems as defined originally. We plan a more systematic discussion of this in a later paper.

§ 2. Elementary formal systems

We define the variation of [4] that we will be using.

We suppose available an unlimited supply of *k-place predicate symbols* for each $k > 0$. The other two symbols of our alphabet are an *arrow* and a *comma*. We will be using axiom schemas, so we have no variables in the language itself.

Let ω be the set of natural numbers (including 0). By an *atomic formula* we mean an expression of the form $Pv_1, \ldots, v_k$ where $v_1, \ldots, v_k \in \omega$ and P is a k-place predicate symbol. For convenience we may write $P\boldsymbol{v}$ for $Pv_1, \ldots, v_k$ sometimes. We also define a *pseudo-atomic formula* to be anything of the form $Px_1, \ldots, x_k$ where each x_i is in ω or is a variable. Pseudo atomic formulas are expressions of the metalanguage only.

The notion of *formula* is defined recursively by:

1) an atomic formula is a formula.

2) If X and Y are formulas, so is $X \to Y$.

Formulas are to be thought of as being associated to the right. Thus $A \to B \to C \to D$ should be read as if it were $A \to (B \to (C \to D))$ and thought of as saying A, B and C together imply D.

The metalinguistical notion of *pseudo-formula* is defined in the obvious way. And an *instance* of a pseudo-formula is defined to be the result of replacing all variables by numbers.

By the *conclusion* of a (pseudo) formula we mean the final (pseudo) atomic part of it. Thus, if A is (pseudo) atomic, A is the conclusion of both $X \to A$ and of A itself.

We assume a two-place relation symbol, say S, has been set aside to represent the successor relation on ω. But we will often write $y = x^+$ instead of Sx, y, to make reading easier. Let S^* consist of all atomic formulas of the form Sx, y where y is the successor of x.

We say a pseudo-formula X is *allowable* if S, the successor predicate symbol, does not occur in the conclusion of X.

Let $\mathfrak{A}$ be a finite set of pseudo-formulas, each allowable. By a *derivation* from $\mathfrak{A}$ we mean a finite sequence of formulas such that each term of the sequence either: 1) is a member of S^*, or 2) is an instance of some member of $\mathfrak{A}$, or 3) comes from two earlier terms by the rule

$$\text{(MP)} \qquad \frac{X \quad X \to Y}{Y} \quad \text{provided } X \text{ is atomic.}$$

If there is such a derivation ending with X, we say X is derivable from $\mathfrak{A}$.

In this way $\mathfrak{A}$ determines a simple deductive system, called an *elementary formal system.* We call the members of $\mathfrak{A}$ the *axiom schemas* for the elementary formal system they determine.

Let P be a k-place predicate symbol, and $\mathfrak{P} \subseteq \omega^k$. We say P *represents* $\mathfrak{P}$ *in the elementary formal system determined by* $\mathfrak{A}$ if

$$\boldsymbol{v} \in \mathfrak{P} \text{ iff } P\boldsymbol{v} \text{ is derivable from } \mathfrak{A}.$$

We say $\mathfrak{P}$ is *representable using the axiom schemas* $\mathfrak{A}$ if there is some predicate symbol which represents $\mathfrak{P}$ in the elementary formal system determined by $\mathfrak{A}$. Finally, $\mathfrak{P}$ is *representable* if it is representable using some finite set of axiom schemas $\mathfrak{A}$.

The following has a straightforward demonstration.

Theorem. *A relation* $\mathfrak{R}$ *on* ω *is representable iff* $\mathfrak{R}$ *is recursively enumerable* (*in any standard sense*).

§ 3. ω-elementary formal systems

We add an infinite-premise rule of derivation to the machinery of elementary formal systems as just described.

First, modify the alphabet by adding the additional symbol $\forall$. Now an atomic formula is a string $Px_1, \ldots, x_k$ where P is k-place and each x_i either is in ω, or is $\forall$. (Similarly modify the notion of pseudo-atomic formula.) Formulas are built up as before, but from the enlarged class of atomic formulas. Otherwise no syntactical changes are made; *instance* still means *numerical* instance, for example.

Intuitively, $P\boldsymbol{v}, \forall, \boldsymbol{w}$ is to mean $P\boldsymbol{v}, n, \boldsymbol{w}$ holds for each number n. Now we give rules governing the formal use of $\forall$. This can be done in two different but equivalent ways.

Version 1. We make the *restriction* that $\forall$ may not occur in the conclusion of any axiom schema. And we add the following rule of derivation.

ω-rule. If $P\boldsymbol{v}, \forall, \boldsymbol{w}$ is atomic, then

$$\frac{P\boldsymbol{v}, n, \boldsymbol{w} \text{ for each } n \in \omega}{P\boldsymbol{v}, \forall, \boldsymbol{w}}$$

Version 2. This time no restriction is placed on the form of axiom schemas. But we add two new rules of derivation, the ω-rule as in version 1, and also the

Inverse ω-rule. If $P\boldsymbol{v}, \forall, \boldsymbol{w}$ is atomic, then

$$\frac{P\boldsymbol{v}, \forall, \boldsymbol{w}}{P\boldsymbol{v}, n, \boldsymbol{w}} \quad \text{for each } n \in \omega.$$

The notion of an *ω-elementary formal system* is formulated analogously to the notions in § 2, but derivations are now well-ordered (possibly) infinite sequences, allowing (version 1 or version 2) of the above rules. Call a relation $\mathfrak{R}$ *ω-r.e.* if it is representable in some ω-elementary formal system. Call $\mathfrak{R}$ *ω-recursive* if $\mathfrak{R}$ and its complement are both ω-r.e. In the next two sections we show

Theorem A. *Using either version, for any relation $\mathfrak{R}$ on ω:*
1) *$\mathfrak{R}$ is ω-r.e. iff $\mathfrak{R}$ is Π^1_1,* 2) *$\mathfrak{R}$ is ω-recursive iff $\mathfrak{R}$ is hyperarithmetic.*

§ 4. Kleene's $\mathfrak{O}$:

We show $\mathfrak{O}$, the set of Kleene's ordinal notations, is ω-r.e., and obtain half of theorem A as a consequence. We take $\mathfrak{O}$ as formulated in [1].

We need the following fact from ordinary recursion theory. Let f_i be the partial recursive one-place function with (function) index i. Let U be the three-place relation defined by

$$Uz, x, y \Leftrightarrow f_z(x) = y.$$

Then U is r.e. We also need the following useful fact: any relation which is r.e. is ω-r.e. using the same axiom schemas.

Now we introduce the axiom schemas for $\mathfrak{O}$. We do this in groups, first explaining what each new predicate symbol is to represent.

Begin with axiom schemas for the r.e. relations U, multiplication and exponentiation. For reading ease we write $y = x \cdot z$ instead of the relation form Mx, y, z, say, and similarly for exponentiation. Now

$Dx, y :\Leftrightarrow x$ is in the domain of f_y

$$Uy, x, w \to Dx, y;$$

$Rx, y :\Leftrightarrow x$ is in the range of f_y

$$Uy, w, x \to Rx, y;$$

$Tx :\Leftrightarrow f_x$ is total

$$D\forall, x \to Tx;$$

$<_o$: the ordering of Kleene's $\mathfrak{O}$

$$1 <_o 2, \quad 1 <_o y \to z = 2^y \to y <_o z$$

[more axioms for $<_o$ presently];

$Gn, y :\Leftrightarrow f_y(n) <_o f_y(n+1)$

$$Uy, n, z \to k = n^+ \to Uy, k, w \to z <_o w \to Gn, y;$$
$$Ty \to G\forall, y \to Rz, y \to a = 5^y \to b = 3 \cdot a \to z <_o b;$$

$$x <_o y \to y <_o z \to x <_o z;$$

O: Kleene's $\mathfrak{O}$

$$x <_o y \to Ox; \quad x <_o y \to Oy.$$

It is not hard to see that, using either version, O does represent $\mathfrak{O}$ in the ω-elementary formal system with the above axiom schemas. Thus

Lemma 4.1. *Kleene's $\mathfrak{O}$ is ω-r.e. (using either version).*

Lemma 4.2. *If $Q \leqq_1 P$, and P is ω-r.e. then Q is also ω-r.e. (using either version).*

Proof. If $Q \leqq_1 P$, then for some one-one recursive f, $x \in Q \Leftrightarrow f(x) \in P$. Now the relation $y = f(x)$ is r.e. So, for axioms for Q, simply take:

axioms for P, axioms for $y = f(x)$, $y = f(x) \to Py \to Qx$.

Theorem. *Under either version, every Π^1_1 relation is ω-r.e.*

Proof. Kleene's $\mathfrak{O}$ is a complete Π^1_1 set ([2], pg. 397), hence if $\mathfrak{R}$ is any Π^1_1 relation, $\mathfrak{R} \leqq_1 \mathfrak{O}$. The result follows by lemmas 4.1 and 4.2.

§ 5. Models

We introduce the notion of a model for an ω-elementary formal system, out of which will come the other half of theorem A. The idea is essentially that of R-definability in [3].

Let $\mathfrak{A}$ be a set of axiom schemas, involving, besides the successor predicate symbol, the predicate symbols $P_1, \ldots, P_k$. A *model structure* for $\mathfrak{A}$ is a k-tuple $\langle \mathfrak{P}_1, \ldots, \mathfrak{P}_k \rangle$ where, if P_i is n-place then $\mathfrak{P}_i \subseteq \omega^n$.

Next we define the notion of *truth in a model structure* (for formulas involving only successor, $P_1, \ldots, P_k$). First, for atomic formulas.

a) $y = x^+$ is true in $\langle \mathfrak{P}_1, \ldots, \mathfrak{P}_k \rangle$ if, in fact, y is the successor of x.
b) Suppose P_i is n-place, and $\boldsymbol{v} \in \omega^n$. Then $P_i\boldsymbol{v}$ is true in $\langle \mathfrak{P}_1, \ldots, \mathfrak{P}_k \rangle$ if $\boldsymbol{v} \in \mathfrak{P}_i$.
c) $P_i\boldsymbol{v}, \forall, \boldsymbol{w}$ is true in $\langle \mathfrak{P}_1, \ldots, \mathfrak{P}_k \rangle$ if $P_i\boldsymbol{v}, n, \boldsymbol{w}$ is true in $\langle \mathfrak{P}_1, \ldots, \mathfrak{P}_k \rangle$ for each $n \in \omega$.

For non-atomic formulas, call $X_1 \to X_2 \to \ldots \to X_q$ true in $\langle \mathfrak{P}_1, \ldots, \mathfrak{P}_k \rangle$ (where $X_1, X_2, \ldots, X_q$ are atomic) if one of $X_1, \ldots, X_{q-1}$ is not true, or X_q is true.

We say $\langle \mathfrak{P}_1, \ldots, \mathfrak{P}_k \rangle$ is a *model* for $\mathfrak{A}$ if it is a model structure for $\mathfrak{A}$ and each instance of a member of $\mathfrak{A}$ is true in $\langle \mathfrak{P}_1, \ldots, \mathfrak{P}_k \rangle$.

Lemma 5.1. *If $\langle \mathfrak{P}_1, \ldots, \mathfrak{P}_k \rangle$ is a model for $\mathfrak{A}$ then any formula derivable (under either version) in the ω-elementary formal system with axiom schemas $\mathfrak{A}$ is true in $\langle \mathfrak{P}_1, \ldots, \mathfrak{P}_k \rangle$.*

Proof. By transfinite induction on proof length.

Lemma 5.2. *Let $P_i\boldsymbol{v}$ be an atomic formula not derivable from $\mathfrak{A}$. Then there is a model $\langle \mathfrak{P}_1, \ldots, \mathfrak{P}_k \rangle$ for $\mathfrak{A}$ in which $P_i\boldsymbol{v}$ is not true (under either version).*

Proof. Suppose P_j is n-place. Set $\mathfrak{P}_j = \{\boldsymbol{v} \in \omega^n \mid P_j\boldsymbol{v}$ is derivable from $\mathfrak{A}\}$.

Now it can be checked that $\langle \mathfrak{P}_1, \ldots, \mathfrak{P}_k \rangle$ is a model for $\mathfrak{A}$ in which $P_i\boldsymbol{v}$ is not true.

The above two lemmas give us

Theorem 5.3. *Let $P_1\boldsymbol{v}$ be an atomic formula. Then, under either version, $P_i\boldsymbol{v}$ is derivable from $\mathfrak{A}$ iff $P_i\boldsymbol{v}$ is true in every model for $\mathfrak{A}$.*

Lemma 5.4. *That $\langle \mathfrak{P}_1, \ldots, \mathfrak{P}_k \rangle$ is a model for $\mathfrak{A}$ is arithmetic in $\mathfrak{P}_1, \ldots, \mathfrak{P}_k$.*

Proof. We content ourselves with an illustrative example. Say $k = 2$, and $\mathfrak{A}$ consists of the two schemas

$$y = x^+ \to P_1x, y, 3,\quad P_1x, \forall, 3 \to P_2, x, 4.$$

Then an appropriate formula is

$$\varphi(\mathfrak{P}_1, \mathfrak{P}_2) = (\forall x)\,(\forall y)\,[y = x^+ \supset \mathfrak{P}_1(x, y, 3)] \wedge (\forall x)\,[(\forall z)\,\mathfrak{P}_1(x, z, 3) \supset \mathfrak{P}_2(x, 4)].$$

Lemma 5.5. *That $P_i\boldsymbol{v}$ is true in every model for $\mathfrak{A}$ is Π^1_1.*

Proof. Let $\varphi(\mathfrak{P}_1, \ldots, \mathfrak{P}_k)$ be the formula of lemma 5.4 saying $\langle \mathfrak{P}_1, \ldots, \mathfrak{P}_k \rangle$ is a model for $\mathfrak{A}$. Then, use the formula $(\forall \mathfrak{P}_1) \ldots (\forall \mathfrak{P}_k)\,[\varphi(\mathfrak{P}_1, \ldots, \mathfrak{P}_k) \supset \mathfrak{P}_i\boldsymbol{v}]$.

Lemma 5.5 and theorem 5.3 immediately give

Theorem. *Under either version, every ω-r.e. relation is Π^1_1.*

We have now established theorem A part 1. Part 2 is immediate since $\mathfrak{R}$ is hyperarithmetic iff $\mathfrak{R}$ and its complement are Π^1_1.

§ 6. Definability Results

Let L be the first order language containing: a constant symbol for each number, a relation symbol for the successor relation, and other relation symbols.

Let X be a formula of L. By a *standard model* for X we mean a model in which

1) the domain is ω,
2) the successor relation symbol is interpreted by the successor relation,
3) each number constant is interpreted by the corresponding number,
4) every instance of X is true.

For convenience, since standard models are all we will be considering, let us use the numbers themselves as the constant symbols of L, and for the relation symbols we will use those of elementary formal systems.

Let X be a formula of L and P be an n-place relation symbol. By $\langle X, P\rangle$ we mean

$$\{\boldsymbol{v} \in \omega^n \mid P\boldsymbol{v} \text{ is true in every standard model for } X\}.$$

We say a relation $\mathfrak{R}$ on ω is *characterizable* if, for some X and some P, $\mathfrak{R} = \langle X, P\rangle$.

Theorem B.

1) *The characterizable relations are exactly the ω-r.e. (or Π_1^1) relations.*

2) *The relations characterizable using formulas of L with no quantifiers are exactly the r.e. relations.*

Proof of part 1. Suppose $\mathfrak{R}$ is ω-r.e. Then $\mathfrak{R}$ is representable in some ω-elementary formal system, say (to give a familiar example) by P_2, using the axiom schemas $\mathfrak{A}$, where $\mathfrak{A}$ consists of

$$y = x^+ \rightarrow P_1x, y, 3, \quad P_1x, \forall, 3 \rightarrow P_2x, 4.$$

Then take X to be the formula of L:

$$[y = x^+ \supset P_1x, y, 3] \wedge [(\forall w)\, P_1x, w, 3 \supset P_2x, 4].$$

It is easy to see the standard models for X are just the models for $\mathfrak{A}$ as defined in § 5. It follows that $\mathfrak{R} = \langle X, P_2\rangle$.

Conversely, suppose $\mathfrak{R} = \langle X, P\rangle$ for some X and some P. Then

$$\boldsymbol{v} \in \mathfrak{R} \text{ iff } P\boldsymbol{v} \text{ is true in all standard models for } X.$$

But the proofs of lemmas 5.4 and 5.5 can easily be modified to show this is a Π_1^1 notion.

This completes the proof of part 1.

Remark. Actually a stronger result has been shown: The ω-r.e. relations are characterizable using formulas of L of a certain standard form, namely, conjunctions of formulas of the form $(X_1 \wedge X_2 \wedge \ldots \wedge X_{q-1}) \supset X_q$ where X_q is atomic and each of $X_1, \ldots, X_{q-1}$ is either atomic, or atomic with some universal quantifiers prefixed. Formulas of this form arise naturally out of ω-elementary formal systems using version 1. This result can be further strengthened, to restrict the variety of predicate symbols used, but we do not pursue this here.

Proof of part 2. If $\mathfrak{R}$ is r.e., it is representable in some elementary formal system, say by P, using axiom schemas $\mathfrak{A}$. Now proceed just as in the proof of part 1. Since the symbol $\forall$ is not used in $\mathfrak{A}$, the resulting formula X, such that $\mathfrak{R} = \langle X, P\rangle$, will be quantifier free.

Conversely, suppose $\mathfrak{R} = \langle X, P\rangle$ where X has no quantifiers. Let X_I be the set of instances of X, together with all true instances of the successor relation.

Auxillary Lemma. *Writing $\vdash$ to denote the consequence relation of propositional logic: Pv is true in all standard models for X iff $X_I \vdash Pv$.*

Assuming the truth of this lemma, we complete the proof of part 2.

Let A^0 be the Gödel number of the formula A (under some standard Gödel numbering), and let $X_I^0 = \{A^0 \mid A \in X_I\}$. X_I^0 is easily seen to be r.e., hence $C = \{A^0 \mid X_I \vdash A\}$ is r.e. It follows that $\{v \mid (Pv)^0 \in C\}$ is also r.e. But $\{v \mid (Pv)^0 \in C\} = \{v \mid X_I \vdash Pv\} = \mathfrak{R}$ by the lemma.

Proof of auxillary lemma. If $X_I \vdash A$, A is true in any model for X_I. But a standard model for X is also a model for X_I, hence if $X_I \vdash Pv$, Pv is true in any standard model for X.

Conversely, suppose not $X_I \vdash Pv$. Then $X_I \cup \{\sim Pv\}$ is consistent. Extend it to a maximal consistent set M. Define a model by:

setting domain $= \omega$,

interpreting the successor relation symbol to be the successor relation,

interpreting P_i by $\{v \mid P_i v \in M\}$,

interpreting each number constant as that number.

In this model, all members of M are true. So all of X_I is true in it. It follows that it is a standard model for X, and in it, Pv is not true.

Remark. Again, a stronger result has actually been established. The r.e. relations are characterizable using quantifier free formulas which are conjunctions of formulas of the form

$$(X_1 \wedge \ldots \wedge X_{q-1}) \supset X_q, \quad \text{where each } X_i \text{ is atomic.}$$

This is essentially the result of [3].

References

[1] DAVIS, M., Computability and Unsolvability. McGraw Hill 1958.
[2] ROGERS JR., H., Theory of Recursive Functions and Effective Computability. McGraw-Hill 1967.
[3] SMULLYAN, R., On Definability by recursion (abstract). A.M.S. Bulletin 62 (1956), 601.
[4] SMULLYAN, R., Theory of Formal Systems. Revised edition, Princeton University Press 1961.

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