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A GENERALIZATION OF ELEMENTARY FORMAL SYSTEMS

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1. Introduction

In [6] Smullyan gave an elegant development of recursion theory based on elementary formal systems. These dealt directly with words over a finite alphabet, and only indirectly with numbers, via "names" for them. We generalize the notion of elementary formal system, by separating "structural properties" from "subject matter." The result provides a natural "recursion theory" for any structure, words and numbers being particular examples.

Our notion of recursion theory over the natural numbers can be turned into hyperarithmetic theory by the addition of a simple infinitary rule (an ω-rule) [1]. We formulate the rule so that it applies to all our recursion theories, turning them into what we call ω-recursion theories. For both recursion and ω-recursion theories we define a natural generalization of enumeration operator. We investigate the structural characteristics of these operators, and prove an analog of the First Recursion Theorem for them.

2. Elementary formal systems

Let $\mathscr{A}$ be an infinite set, and let $\mathscr{R}_1, \ldots, \mathscr{R}_k$ be relations on $\mathscr{A}$. We cal $k+1$ tuple $\langle \mathscr{A}, \mathscr{R}_1, \ldots, \mathscr{R}_k \rangle$ a *structure*. We allow trivial structures $\langle \mathscr{A} \rangle$ We set up a simple logical calculus relative to a particular structure, so for the rest of this section, let $\mathfrak{A} = \langle \mathscr{A}, \mathscr{R}_1, \ldots, \mathscr{R}_k \rangle$ be a fixed structure.

We suppose available an unlimited supply of n-place predicate symbols for each $n > 0$. We informally use P, Q, R, etc. to represent them. The other two symbols of our alphabet are an *arrow* and a *comma*. We will use axiom schemas, so variables are not needed in the language itself, and we need no rule of substitution.

By an *atomic formula* we mean an expression of the form $Pv_1, \ldots, v_n$ where $v_1, \ldots, v_n \in \mathscr{A}$ and P is an n-place predicate symbol. For convenience we may write $P\boldsymbol{v}$ for $Pv_1, \ldots, v_n$. We also define a *pseudo-atomic formula* to be anything of the form $Px_1, \ldots, x_n$ where each x_i is in $\mathscr{A}$ or is a variable. Pseudo-atomic formulas are expressions of the metalanguage only.

The notion of *formula* is defined by the following rules:

1) an atomic formula is a formula,

2) if $X_1, X_2, \ldots, X_n$ are formulas, so is $X_1 \to X_2 \to \ldots \to X_n$. Formulas are to be thought of as being associated to the right. Thus $A \to B \to C \to D$ should be read as if it were $A \to (B \to (C \to D))$ and thought of as saying A, B and C together imply D.

The metalinguistical notion of *pseudo-formula* is defined analogously, being built up from pseudo-atomic formulas. And the notion of an *instance* of a pseudo-formula (over $\mathscr{A}$) also has an obvious definition. Any instance of a pseudo-formula is a formula.

By the *conclusion* of a (pseudo) formula we mean the final (pseudo) atomic part of it. Thus if A is (pseudo) atomic, A is the conclusion of $X \to A$, and also of A itself.

Let $R_1, \ldots, R_k$ be distinct predicate symbols permanently assigned to the relations $\mathscr{R}_1, \ldots, \mathscr{R}_k$, such that R_j is n-place if $\mathscr{R}_j$ is an n-ary relation. Let $\mathscr{R}_j^*$ consist of all atomic formulas of the form $R_j\boldsymbol{v}$ where $\mathscr{R}_j\boldsymbol{v}$ is true.

We say a pseudo-formula X is *allowable* if none of $R_1, \ldots, R_k$ occurs in the conclusion of X.

Let $\{A_1, \ldots, A_m\}$ be some finite set of pseudo-formulas, each allowable. By a *derivation from* $\{A_1, \ldots, A_m\}$ (over $\mathfrak{A}$) we mean a finite sequence of formulas, $X_1, \ldots, X_j$ such that each term of the sequence either

1) is a member of $\mathscr{R}_1^* \cup \ldots \cup \mathscr{R}_k^*$ or
2) is an instance of some A_i or
3) comes from two earlier terms by the rule

MP $$\frac{X \quad X \to Y}{Y} \text{ provided } X \text{ is atomic.}$$

If there is a derivation ending with X, we say *X is derivable from* $\{A_1, \ldots, A_m\}$ (over $\mathfrak{A}$).

$\{A_1, \ldots, A_m\}$ determines, relative to $\mathfrak{A}$, a simple deductive system, called an *elementary formal system* (over $\mathfrak{A}$). Each A_i is an *axiom* for that elementary formal system.

Let P be a p-place predicate symbol, and $\mathscr{P} \subseteq \mathscr{A}^p$. We say *$P$ represents $\mathscr{P}$ in the elementary formal system determined by* $\{A_1, \ldots, A_m\}$ if $\boldsymbol{v} \in \mathscr{P}$ iff $P\boldsymbol{v}$ is derivable from $\{A_1, \ldots, A_m\}$.

We say $\mathscr{P}$ *is representable in the elementary formal system determined by* $A_1, \ldots, A_m$, if there is some predicate symbol P which represents $\mathscr{P}$ in that elementary formal system.

Finally, we say $\mathscr{P}$ *is recursively enumerable* (*r.e.*) (over $\mathfrak{A}$) if $\mathscr{P}$ is representable in some elementary formal system (over $\mathfrak{A}$). Also $\mathscr{P}$ is *recursive* (over $\mathfrak{A}$) if both $\mathscr{P}$ and $\mathscr{A}^p - \mathscr{P}$ are r.e. (over $\mathfrak{A}$).

3. Examples

EXAMPLE 1. $\mathscr{A} = \omega$. $\mathscr{R}$ is the successor relation: $\mathscr{R}x, y$ iff $y = x+1$. Let $\mathfrak{A}_N$ be the structure $\langle \mathscr{A}, \mathscr{R} \rangle$. It can be shown that a relation is r.e. over $\mathfrak{A}_N$ iff it is r.e. in the usual sense.

EXAMPLE 2. $\mathscr{A}$ is the set of signed integers. $\mathscr{R}$ is the successor relation for signed integers. Let $\mathfrak{A}_Z$ be the structure $\langle \mathscr{A}, \mathscr{R} \rangle$. A relation is r.e. over $\mathfrak{A}_Z$ iff under any standard Gödel numbering of the signed integers, it corresponds to an r.e. set of natural numbers.

EXAMPLE 3. $\mathscr{A} = \omega \times \omega$. We use two relations on $\mathscr{A}$. $\mathscr{R}_1 x, y$ iff the first component of y is the successor of the first component of x. $\mathscr{R}_2$ is similarly the second component successor relation. Let $\mathfrak{A}_Q$ be the structure $\langle \mathscr{A}, \mathscr{R}_1, \mathscr{R}_2 \rangle$. A relation is r.e. over $\mathfrak{A}_Q$ iff the result of applying a recursive pairing function to its members produces a relation r.e. in the conventional sense.

EXAMPLE 4. $\mathscr{A} = \text{H.F.} = R_\omega$. $\mathscr{R}_1$ is the union relation, $\mathscr{R}_1(x, y, z)$ iff $x \cup y = z$. $\mathscr{R}_2$ is the unordered pair relation, $\mathscr{R}_2(x, y, z)$ iff $\{x, y\} = z$. Let $\mathfrak{A}_S = \langle \mathscr{A}, \mathscr{R}_1, \mathscr{R}_2 \rangle$. A relation is r.e. over $\mathfrak{A}_S$ iff it is Σ over H. F. *Remark*: This example is essentially unchanged if we use the single relation $x \cup \{y\} = z$.

EXAMPLE 5. $\mathscr{A}$ is the set of words over a fixed finite alphabet. $\mathscr{R}$ is the concatenation relation on $\mathscr{A}$. Let $\mathfrak{A}_W$ be the structure $\langle \mathscr{A}, \mathscr{R} \rangle$. Elementary formal systems over $\mathfrak{A}_W$ are those called *pure* in [6]. *Remark*: This example too is essentially unchanged if, instead of $\mathscr{R}$ we use one relation for each letter a of our finite alphabet, taking, say, $\mathscr{R}_a\, x, y$ iff y is word x with a added to the end.

EXAMPLE 6. $\mathscr{A}$ is the set of real numbers. Take the two relations of addition and order for reals. In the resulting structure, for example, the set of integers is r.e. Also, the function $f(x) = [x]$, mapping each real x to the greatest integer $\leqslant x$, is recursive (as a relation).

EXAMPLE 7. Let $\mathfrak{A}$ be the structure $\langle F \cup V; F, +_F, \times_F, V, +_V \times_S, ^* \rangle$ where F is a field under the operations $+_F$ and $\times_F$, V is an n-dimensional vector space over F, with vector addition $+_V$ and scalar multiplication

$\times_S$, and $*$ is an inner product on V. Call a function *partial recursive* if its graph is r.e. The Gram–Schmidt orthogonalization process can be "captured" by an elementary formal system over this structure to show: there is a partial recursive function f, taking n-tuples of vectors to n-tuples of vectors, such that if $f(x_1, \ldots, x_n) = \langle y_1, \ldots, y_n \rangle$ then $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ span the same subspace of V, and $\{y_1, \ldots, y_n\}$ is an orthogonal set (some of the y_i may be 0).

4. Enumeration operators

We modify elementary formal systems so that they may accept inputs as well as produce outputs.

Some notation. Suppose $\mathfrak{A}$ is the structure $\langle \mathscr{A}, \mathscr{R}_1, \ldots, \mathscr{R}_k \rangle$ and $\mathscr{R}$ is some relation on $\mathscr{A}$. We write $\langle \mathfrak{A}, \mathscr{R} \rangle$ for the structure $\langle \mathscr{A}, \mathscr{R}_1, \ldots, \mathscr{R}_k, \mathscr{R} \rangle$. Suppose E is a set of axioms for an elementary formal system over $\mathfrak{A}$, and A is a predicate symbol. We write $E \vdash_{\mathfrak{A}} Ax$ to mean there is a derivation of Ax from E over $\mathfrak{A}$. Then $\{x \mid E \vdash_{\mathfrak{A}} Ax\}$ is the relation which A represents in the elementary formal system (with axioms) E over $\mathfrak{A}$.

Now, let $\mathfrak{A}$ be some structure $\langle \mathscr{A}, \mathscr{R}_1, \ldots, \mathscr{R}_k \rangle$, fixed for the rest of this section. Suppose $\mathscr{P}$ is some n-place relation on $\mathscr{A}$. Let E be an elementary formal system over the structure $\langle \mathfrak{A}, \mathscr{P} \rangle$ in which, say, the predicate symbol P has been assigned to $\mathscr{P}$. Let Q be an m-place predicate symbol. Using the axioms E, Q represents a certain m-ary relation on $\mathscr{A}$. Now suppose we keep E fixed, but change $\mathscr{P}$ to $\mathscr{P}'$, still using P to represent it. Then Q will represent a different relation on $\mathscr{A}$. In this way a certain operator on $\mathscr{A}$ is created, which we may symbolize by $[E_Q^P]$. It uses the axioms E, takes whatever P represents as input, and gives whatever Q represents as output. Formally

$$[E_Q^P](\mathscr{P}) = \{x \mid E \vdash_{\langle \mathfrak{A}, \mathscr{P} \rangle} Qx\}$$

[where the predicate symbol P is assigned to $\mathscr{P}$].

We call the maps $[E_Q^P]$ *enumeration operators* over the structure $\mathfrak{A}$.

Let $\mathfrak{A}_N$ be the structure of arithmetic as defined in Example 1 of the last section. It can be shown that the enumeration operators over $\mathfrak{A}_N$ coincide with enumeration operators as defined in [5].

Let $[E_Q^P]$ be an enumeration operator over the structure $\mathfrak{A}$ taking n-ary relations to m-ary relations. That is, P is n-place and Q is m-place. We call pair $\langle n, m \rangle$ the *order* of $[E_Q^P]$. Of course, $n, m > 0$.

5. The ω-rule

We add an infinite-premise rule of derivation to the machinery of elementary formal systems as defined above.

First, we modify the alphabet by adding the additional symbol $\forall$.

Now an atomic formula is a string $Px_1, \ldots, x_n$ where P is n-place and each x_i either is in $\mathscr{A}$, or is $\mathcal{V}$. (We similarly modify the notion of pseudo-atomic formula, and formula.) Otherwise no syntactical changes are made; *instance* still means instance over $\mathscr{A}$; for example.

Intuitively, $P\boldsymbol{v}, \mathcal{V}, \boldsymbol{w}$ is to mean $P\boldsymbol{v}, a, \boldsymbol{w}$ holds for each $a \in \mathscr{A}$. Now we give rules governing the formal use of $\mathcal{V}$. We make the *restriction* that $\mathcal{V}$ may not occur in the conclusion of any axiom schema. And we add one more rule of derivation.

ω-rule: If $P\boldsymbol{v}, \mathcal{V}, \boldsymbol{w}$ is atomic, then

$$\frac{P\boldsymbol{v}, a, \boldsymbol{w} \text{ for each } a \in \mathscr{A}}{P\boldsymbol{v}, \mathcal{V}, \boldsymbol{w}}.$$

The notion of an *ω-elementary formal system* is formulated as expected. Derivations are now well ordered (possibly) infinite sequences, allowing ω-rule applications. Call a relation $\mathscr{P} \subseteq \mathscr{A}^p$ *ω-r.e.* over $\mathfrak{A}$ if it is representable in some ω-elementary formal system over $\mathfrak{A}$. Also $\mathscr{P}$ is ω-recursive if both $\mathscr{P}$ and $\mathscr{A}^p - \mathscr{P}$ are ω-r.e.

Let $\mathfrak{A}_N$ be the structure of arithmetic (Example 1 from § 3). In [1] is a direct proof that, for $\mathfrak{A}_N$, ω-r.e. is the same as Π_1^1 and ω-recursive is the same as hyperarithmetic. More generally, in any structure, ω-recursive coincides with hyperelementary, as defined in [4].

The definition of enumeration operator over $\mathfrak{A}$ may be modified in the obvious way to define the notion of *ω-enumeration operator* over $\mathfrak{A}$. We skip the details.

6. Basic structural properties

The collection of enumeration operators over a structure $\mathfrak{A} = \langle \mathscr{A}, \mathscr{R}_1, \ldots, \mathscr{R}_k \rangle$ and the collection of ω-enumeration operators over $\mathfrak{A}$ have certain common structural features, which we now develop. We note that the ω-rule plays no role in many of our proofs below, so we can treat recursion and ω-recursion theories simultaneously.

THEOREM 6.1. *Suppose I and J are (ω) enumeration operators over $\mathfrak{A}$, I is of order $\langle n, m \rangle$ and J is of order $\langle m, p \rangle$. Then the composition $J \circ I$ is also an (ω) enumeration operator over $\mathfrak{A}$, of order $\langle n, p \rangle$.*

Proof. Say $I = [E_B^A]$ and $J = [E_D'^C]$. We may suppose without loss of generality that E and E' contain no predicate symbols in common other than $R_1, \ldots, R_k$. (Call E and E' *disjoint* if this is the case.) Consider the (ω) enumeration operator $[F_D^A]$ where F consists of

the axioms of E,
the axioms of E',
$B\boldsymbol{x} \to C\boldsymbol{x}$.

It is easy to see that $J \circ I = [F_D^A]$.

DEFINITION. Let I and J be (ω) enumeration operators over $\mathfrak{A}$, both of order $\langle n, m\rangle$. We define two maps as follows:

$$(I\cap J)(\mathscr{P}) = I(\mathscr{P})\cap J(\mathscr{P}),$$
$$(I\cup J)(\mathscr{P}) = I(\mathscr{P})\cup J(\mathscr{P}).$$

THEOREM 6.2. *The collection of (ω) enumeration operators over $\mathfrak{A}$ is closed under $\cap$ and $\cup$.*

Proof. Say $I = [E^A_B]$ and $J = [E'^C_D]$ where E and E' are disjoint. Then $I\cap J = [F^G_H]$ and $I\cup J = [F^G_K]$ where F consists of

the axioms of E,
the axioms of E',
$Gx \to Ax$,
$Gx \to Cx$,
$By \to Dy \to Hy$,
$By \to Ky$,
$Dy \to Ky$.

DEFINITION. Let I be an (ω) enumeration operator of order $\langle n, m\rangle$ and let J be an (ω) enumeration operator of order $\langle n, m'\rangle$ over $\mathfrak{A}$. By $I\times J$ we mean the map of order $\langle n, m+m'\rangle$ given by

$$(I\times J)(\mathscr{P}) = I(\mathscr{P})\times J(\mathscr{P}).$$

THEOREM 6.3. *The collection of (ω) enumeration operators over $\mathfrak{A}$ is closed under $\times$.*

DEFINITION. P^n is the *projection operator of order* $\langle n, n-1\rangle$ defined by

$$P^n(\mathscr{P}) = \{\langle x_1, \ldots, x_{n-1}\rangle \mid (\exists y)\langle y, x_1, \ldots, x_{n-1}\rangle \in \mathscr{P}\}.$$

D^n is the *dual projection operator of order* $\langle n, n-1\rangle$ defined by

$$D^n(\mathscr{P}) = \{\langle x_1, \ldots, x_{n-1}\rangle \mid (\forall y)\langle y, x_1, \ldots, x_{n-1}\rangle \in \mathscr{P}\}.$$

THEOREM 6.4. 1) *P^n is both an enumeration and an ω-enumeration operator over $\mathfrak{A}$.*

2) *D^n is an ω-enumeration operator over $\mathfrak{A}$.*

DEFINITION. Let S be a map from $\mathscr{A}^n$ to $\mathscr{A}^k$. S is a *sequential operator* if, for every $j \leqslant k$, either

1) there is an $i \leqslant n$ such that for every $\boldsymbol{x} \in \mathscr{A}^n$, the jth term of $S(\boldsymbol{x})$ is the ith term of $\boldsymbol{x}$, or

2) there is some $c \in \mathscr{A}$ such that for every $\boldsymbol{x} \in \mathscr{A}^n$ the jth term of $S(\boldsymbol{x})$ is c.

DEFINITION. Let $S\colon \mathscr{A}^n \to \mathscr{A}^k$ be a sequential operator. By S^{-1}: power set $(\mathscr{A}^k) \to$ power set $(\mathscr{A}^n)$ we mean the map given by

$$S^{-1}(\mathscr{R}) = \{\boldsymbol{v} \in \mathscr{A}^n \mid S(\boldsymbol{v}) \in \mathscr{R}\}.$$

We call such a map an *explicit transformation*.

THEOREM 6.5. *Every explicit transformation is an (ω) enumeration operator.*

THEOREM 6.6. *Each* (ω) *enumeration operator* I *over* $\mathfrak{A}$ *is monotone, that is,* $\mathscr{A} \subseteq \mathscr{B} \Rightarrow I(\mathscr{A}) \subseteq I(\mathscr{B})$.

THEOREM 6.7. *Each enumeration operator* I *over* $\mathfrak{A}$ *is compact, that is,* $\boldsymbol{v} \in I(\mathscr{P})$ *iff for some finite* $\mathscr{F} \subseteq \mathscr{P}$, $\boldsymbol{v} \in I(\mathscr{F})$.

7. The first recursion theorem

THEOREM 7.1. *Let* I *be an* (ω) *enumeration operator of order* $\langle n, n\rangle$. I *has a minimal fixed point which is* (ω) *r.e.*

Proof. I is monotone. Set

$$I^{\alpha} = I\Big(\bigcup_{\beta<\alpha} I^{\beta}\Big).$$

By standard monotone operator arguments (see [4]) $I^{\infty} = \bigcup_{\bullet} I^{\alpha}$ is the minimal fixed point of I. We show it is (ω) r.e. [*Remark*: If I is an enumeration operator, it is compact, in which case I^{∞} is simply I^{ω}.]

Say $I = [E_B^A]$. (We note that, in E, A can not occur in the conclusion of any axiom.) By definition,

$$\boldsymbol{x} \in I(\mathscr{P}) \quad \text{iff} \quad E \vdash_{\langle \mathfrak{A}, \mathscr{P}\rangle} B\boldsymbol{x}.$$

Now let $E' = E \cup \{B\boldsymbol{x} \to A\boldsymbol{x}\}$ (that is, E' is E+"output = input"). We claim B represents I^{∞} using E'. That is,

$$I^{\infty} = \{\boldsymbol{x} \mid E' \vdash_{\mathfrak{A}} B\boldsymbol{x}\}.$$

1) For each α, we show

$$I^{\alpha} \subseteq \{\boldsymbol{x} \mid E' \vdash_{\mathfrak{A}} B\boldsymbol{x}\}$$

by induction on α. Well, suppose

$$I^{<\alpha} \subseteq \{\boldsymbol{x} \mid E' \vdash_{\mathfrak{A}} B\boldsymbol{x}\}$$

[where $I^{<\alpha} = \bigcup_{\beta<\alpha} I^{\beta}$ so that $I^{\alpha} = I(I^{<\alpha})$].

Also suppose $\boldsymbol{a} \in I^{\alpha}$ then $\boldsymbol{a} \in I(I^{<\alpha})$, so, by definition

$$(*) \qquad E \vdash_{\langle \mathfrak{A}, I^{<\alpha}\rangle} B\boldsymbol{a}.$$

By induction hypothesis, for all $\boldsymbol{x} \in I^{<\alpha}$, $E' \vdash_{\mathfrak{A}} B\boldsymbol{a}$. But then, for all $\boldsymbol{x} \in I^{<\alpha}$,

$$(**) \qquad E' \vdash_{\mathfrak{A}} A\boldsymbol{x}.$$

By $(*)$ and $(**)$, since also $E \subseteq E'$,

$$E' \vdash_{\mathfrak{A}} B\boldsymbol{a}.$$

Hence $I^{\alpha} \subseteq \{\boldsymbol{x} \mid E' \vdash_{\mathfrak{A}} B\boldsymbol{x}\}$.

2) $I^{\infty} \subseteq \{\boldsymbol{x} \mid E' \vdash_{\mathfrak{A}} B\boldsymbol{x}\}$.

This is immediate.

3) $\{\boldsymbol{x} \mid E' \vdash_{\mathfrak{A}} B\boldsymbol{x}\} \subseteq I^{\infty}$.

We show this by an induction on the length of derivations from E' in $\mathfrak{A}$.

Suppose: if Bx is derivable from E' in $< a$ steps, then $x \in I^\infty$.

Suppose: Ba is derivable from E' in a steps. We show $a \in I^\infty$.

Let $\mathscr{R}_A = \{x \mid Ax$ occurs before the last line in some (fixed) derivation from E' of $Ba\}$.

Let $\mathscr{R}_B = \{x \mid Bx$ occurs before the last line in this derivation$\}$.

Now, A occurs in the conclusion of only one axiom in E', namely $Bx \to Ax$. It follows that

$$\mathscr{R}_A \subseteq \mathscr{R}_B.$$

Also, by induction hypothesis, $\mathscr{R}_B \subseteq I^\infty$. Hence

$$(*) \qquad \mathscr{R}_A \subseteq I^\infty.$$

Finally, it should be clear that

$$E \vdash_{\langle \mathfrak{A}, \mathscr{R}_A \rangle} Ba.$$

This says

$$a \in I(\mathscr{R}_A).$$

Then by (*), since I is monotone,

$$a \in I(I^\infty) = I.$$

This concludes the proof.

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