Justification Logics and Hybrid Logics

Melvin Fitting
Dept. Mathematics and Computer Science
Lehman College (CUNY), 250 Bedford Park Boulevard West
Bronx, NY 10468-1589
e-mail: melvin.fitting@lehman.cuny.edu
web page: comet.lehman.cuny.edu/fitting

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Abstract

Hybrid logics internalize their own semantics. Members of the newer family of justification logics internalize their own proof methodology. It is an appealing goal to combine these two ideas into a single system, and in this paper we make a start. We present a hybrid/justification version of the modal logic T. We give a semantics, a proof theory, and prove a completeness theorem. In addition, we prove a *Realization Theorem*, something that plays a central role for justification logics generally. Since justification logics are newer and less well-known than hybrid logics, we sketch their background, and give pointers to their range of applicability. We conclude with suggestions for future research. Indeed, the main goal of this paper is to encourage others to continue the investigation begun here.

1 Introduction

Hybrid logics are modal logics that internalize aspects of possible world semantics. They contain in their language special propositional letters, *nominals*, that designate possible worlds, rather like the designation of instants of time in some kinds of temporal logics. The presence of nominals can be exploited to give smooth, uniform treatments of completeness, interpolation, and other results. They are covered quite thoroughly in [1], and also in [5]. Between hybrid logics and justification logics, the hybrid variety has been with us longer and is the better known, so I say no more about them for now.

More recently a quite different family of logics, *justification logics*, has been developed. Rather than semantics, these internalize aspects of proofs. Justification logics began with a logic called LP, in [2]. The name stood for "logic of proofs." LP formed part of a program to provide an arithmetic semantics for intuitionistic propositional logic. As is well-known, intuitionistic logic embeds in S4, a result due to Gödel [9]. The difficulty is, the necessity operator of S4 cannot be thought of as formal provability in arithmetic, because of Gödel's second incompleteness theorem. But S4 necessity can be thought of as an *explicit* provability predicate, an idea due to Gödel [10] but left unpublished, and independently discovered by Artemov. This was the motivation for the creation of LP, in which the modal operator has been replaced with a family of explicit *proof terms*. Artemov showed that S4 embeds in LP, that is, for each theorem of S4, there is a way of replacing necessity operators with explicit proof terms to produce a theorem of LP. This is called a *Realization Theorem*. Indeed, these proof terms internalize actual LP proofs, in a certain sense that will be discussed further below. In turn, LP itself embeds in formal arithmetic, Artemov's *Arithmetic Completeness Theorem.* All this together provides an arithmetic semantics for intuitionistic logic.

Since then it has been understood that proofs are one kind of justification, and there can be others. Further, S4 is not the only modal logic to which these techniques can be applied. By now there is a rich and growing family of *justification logics*, corresponding to K, T, S4, S5, and so on, multimodal/multiagent versions of these, versions with quantifiers over justifications, with quantifiers over objects, and so on. Work continues. The ideas have been particularly fruitful when applied to epistemic logics, since they allow for the manipulation of explicit reasons for knowledge or belief. Justification logics even provide appropriate machinery for the formal treatment of various well-known philosophical problems. Artemov's recent paper, [4], provides a broad summary of justification logics and their applications, and is highly recommended. In addition, a complete, annotated bibliography of work in justification logics is maintained by Roman Kuznets, and can be accessed at `http://kuznets.googlepages.com/justificationlogicbibliography`.

Given all this, it is a natural question whether there might be justification logic versions of hybrid logics. Such things would be of interest because they would, in a sense, internalize both semantics and proofs. In this paper we make a bare beginning on the investigation of the question. We provide a justification logic version of the simplest of the hybrid logics, *basic* hybrid logic, but corresponding to T rather than K, because this simplfies certain parts of the work. In the logic that is presented below there are justification terms and there are nominals, but in the semantics not every possible world may be named by a nominal. I will formulate this logic, a semantics for it, and provide proofs of completeness and a realization theorem. Much more interesting would be justification logic versions of hybrid logics that are complete with respect to *named* models, models in which every possible world is named by a nominal. I do not know how to formulate such a logic. What I hope to do here is stimulate others to work on the problem.

2 Justification Logics

We begin with a quick overview of the basics of justification logics, since these are generally less familiar than hybrid logics. In the next section I give my version of basic hybrid logic, justification style. After this, various fundamental results concerning the logic are proved.

2.1 The Language

The simplest justification logic is known as J; it corresponds to the simplest normal modal logic,K. We begin with the family of *justification terms* for J. These are built up from *justification variables*: $x_1, x_2, \ldots$; and *justification constants*: $c_1, c_2, \ldots$. They are built up using the *operation symbols*: $+$ and $\cdot$, both binary. The operation $\cdot$ is an application operation. The intention is, if t is a justification of $X \supset Y$ and u is a justification of X then $t \cdot u$ is a justification of Y. The operation $+$ combines justifications, $t + u$ justifies whatever t justifies and also whatever u justifies. A justification term is called *closed* if it contains no justification variables.

Formulas are built up from *propositional letters*: $P_1, P_2, \ldots$, and a *falsehood constant*, $\bot$, using $\supset$ in the usual way, together with an additional rule of formation, $t{:}X$ is a formula provided t is a justification term and X is a formula. The intuition is, $t{:}X$ asserts that X is the case and t serves as a justification for it. Other connectives are defined as usual. We assume a tight binding for the colon operator, and omit parentheses when possible. Thus, for example, $t{:}X \supset X$ should be understood as $(t{:}X) \supset X$.

2.2 An Axiom System for J

Here is a list of axiom schemes (though we use the term 'axiom' for short).

Classical Axioms	all tautologies
+ *Axioms*	$t{:}X \supset (t+u){:}X$
	$u{:}X \supset (t+u){:}X$
$\cdot$ *Axiom*	$t{:}(X \supset Y) \supset (u{:}X \supset (t \cdot u){:}Y)$

For rules, of course we have the standard one.

$$\textit{Modus Ponens} \quad \frac{X \qquad X \supset Y}{Y}$$

Finally there is a version of the modal necessitation rule, but only at the atomic level. The idea is that constant symbols serve as justifications for truths that we do not further analyize. Essentially, unanalyzed are our axioms, and our usage of constant symbols themselves. The rule is formulated to allow arbitrary constants to be used, but if desired, *constant specifications* can be brought in to exert some control. This will be discussed shortly.

Iterated Axiom Necessitation If X is an axiom and c_1, c_2, ..., c_n are constants, then $c_1{:}c_2{:}\ldots c_n{:}X$.

Suppose X is an axiom and, say, $c{:}d{:}X$ has been introduced into a proof using the rule above. We think of d as a justification of X, and c as a justification of the fact that d justifies X, and do not examine these claims more deeply.

A *constant specification* $\mathcal{C}$ is a specification of which constants are to be used with which axioms. More precisely, it is a set of formulas of the form $c_1{:}c_2{:}\ldots c_n{:}X$, where each c_i is a constant and X is an axiom. A constant specification is required to be closed under the condition: if $c_1{:}c_2{:}\ldots c_n{:}X \in \mathcal{C}$ then $c_2{:}\ldots c_n{:}X \in \mathcal{C}$. A proof *meets constant specification* $\mathcal{C}$ provided that whenever $c_1{:}c_2{:}\ldots c_n{:}X$ is introduced using the Iterated Axiom Necessitation rule, then this formula is a member of $\mathcal{C}$. A constant specification can be given ahead of time, or can be created during the course of a proof. Various conditions can be imposed on constant specifications. A constant specification is *axiomatically appropriate* if all instances of axiom schemes have proof constants—here *this will always be assumed.* Other conditions play a role for various purposes, but we will not consider them here.

If X is a theorem of a normal modal logic, so is $\Box X$, and this is usually taken as a basic rule. For J there is a provable analog due to Artemov, [3]—indeed the proof amounts to internalizing the structure of J proofs. We state the result here, and give a hybrid version and its proof in Section 4.2.

Theorem 2.1 (Internalization) *If X is a theorem of J, then there is a closed justification term t such that $t{:}X$ is also a theorem. (In fact, t can be produced constructively from any axiomatic proof of X.)*

Justification logics were formulated axiomatically above. Sequent calculus formulations also exist—one is given in [3] for instance—and hence also tableau formulations. Unfortunately, all current versions include a rule that does not obey the subformula principle, and this somewhat limits their uses.

Combinators, in the usual sense, can be introduced as constants. For instance, the K combinator can be represented by a constant symbol k, where $k{:}(A \supset (B \supset A))$ is taken to be a theorem introduced via an Iterated Axiom Necessitation rule. Then the $\cdot$ operation corresponds to application of combinators, and thus combinatory logic embeds into a fragment of justification logic. This was pointed out by Artemov, in [3].

Finally, if we think of J as a logic of belef, justification terms can be used to limit the problem of logical omniscience. We could restrict things to formulas that do not involve justification terms that are 'too complex,' measured in terms of number of symbols, or nesting depth of terms, or some other way. Justification terms provide us with natural machinery for the measurement of complexity.

2.3 Semantics for J

The earliest semantics for justification logics is in [11]. In [8] a possible world semantics was formulated, with the semantics of [11] as a single-world version of it. It is this possible world semantics that we present here.

A model is $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$, where $\mathcal{G}$ and $\mathcal{R}$ are a state set and an accessibility relation, as usual. $\mathcal{V}$ is a *valuation* and maps each propositional letter to a set of states. The new item is $\mathcal{E}$, which is an *admissible justification* function, or *evidence* function. The idea is, $\mathcal{E}$ assigns to each justification term t and each formula X a set of states—those states in which t is considered to be relevant evidence for X. Relevant evidence is not meant to be conclusive evidence. The date of publication listed on the title page of a book is relevent evidence for the date on which the book was published, though it could be in error. The color of the binding of a book would not be relevant evidence for date of publication. Here are the conditions on evidence functions.

Weakening $\mathcal{E}(s, X) \cup \mathcal{E}(t, X) \subseteq \mathcal{E}(s + t, X)$

Application $\mathcal{E}(t, X \supset Y) \cap \mathcal{E}(u, X) \subseteq \mathcal{E}(t \cdot u, Y)$

Constant Specification We say $\mathcal{E}$ *meets constant specification* $\mathcal{C}$ provided, if $c{:}X \in \mathcal{C}$ then $\mathcal{E}(c, X) = \mathcal{G}$.

Now a forcing relation, $\Vdash$, between states and formulas is defined, with much of the definition standard. The usual modal condition is replaced by something more complex.

$$\begin{aligned}
\mathcal{M}, \Gamma \Vdash P &\iff \Gamma \in \mathcal{V}(P), \text{ for } P \text{ a propositional letter} \\
\mathcal{M}, \Gamma &\not\Vdash \bot \\
\mathcal{M}, \Gamma \Vdash X \supset Y &\iff \mathcal{M}, \Gamma \not\Vdash X \text{ or } \mathcal{M}, \Gamma \Vdash Y \\
\mathcal{M}, \Gamma \Vdash t{:}X &\iff \mathcal{M}, \Delta \Vdash X \text{ for all } \Delta \in \mathcal{G} \text{ with } \Gamma \mathcal{R} \Delta \\
&\qquad \text{and } \Gamma \in \mathcal{E}(t, X)
\end{aligned}$$

The final condition says that we have $t{:}X$ at Γ if X is necessary at Γ in the sense of being true at all accessible worlds, and t is an admissible justification for X at Γ.

There is also a stronger version of the semantics. A model $\mathcal{M}$ is said to be *fully explanatory* provided, if $\mathcal{M}, \Delta \Vdash X$ for all $\Delta \in \mathcal{G}$ with $\Gamma \mathcal{R} \Delta$ then there is some justification t such that $\mathcal{M}, \Gamma \Vdash t{:}X$. Loosely, things that are necessary have their reasons.

The following was proved in [8]. Fix a constant specification $\mathcal{C}$ (axiomatically appropriate). A formula X has a proof in J meeting specification $\mathcal{C}$ if and only if X is true at all states of all J models meeting specification $\mathcal{C}$ if and only if X is true at all states of all J models that are fully explanatory and meet specification $\mathcal{C}$.

2.4 The Realization Theorem

If one takes an axiom of J and replaces every justification term occurrence by $\Box$ one gets an axiom of the modal logic K. Likewise the same replacement turns instances of *modus ponens* in J into instances of *modus ponens* in K, and instances of Iterated Axiom Necessitation into iterated instances of the usual necessitation rule. This replacement is called the *forgetful functor*. By the observations just made, the forgetful functor will convert entire proofs in J into proofs in K. Consequently, the forgetful functor maps theorems of J into theorems of K.

In fact, the forgetful functor is a surjective mapping. Every theorem of K is the image of a theorem of J under it. Indeed, one can be more precise yet.

Theorem 2.2 (Realization) *If X is a theorem of K, there is some replacement of $\Box$ occurrences so that negative occurrences are replaced by distinct justification variables, with the result being a theorem of J, using some axiomatically appropriate constant specification.*

The Realization Theorem says that every theorem of K expands into a theorem of J in which reasons for occurrences of modal operators are provided. Negative occurrences of $\Box$, becoming variables, act like inputs while positive occurrences, becoming terms that may involve those variables, act like outputs. Modal theorems have an implicit input/output structure that justification terms make explicit.

The Realization Theorem is due to Artemov. By now there are several proofs. Most are constructive, as was Artemov's original one, in [2]. These proofs extract justification terms from sequent calculus proofs in K. There is also a non-constructive proof that uses the semantics of Section 2.3, and comes from [8]. We will make use of this approach when we come to a hybrid version of J below.

2.5 A Few Other Justification Logics

The justification logic discussed above was J, corresponding to the modal logic K. In fact all of T, K4, S4, S5, and others have justification versions. The one corresponding to S4 was the first to be developed, as part of the project we discussed earlier to provide an arithmetic semantics for intuitionistic logic. It was, and still is, known as LP. The other justification logics are known as JT, JK4, and JS5. Here is a brief sketch of them.

For JT, LP, and JS5, one adds an axiom scheme $t{:}X \supset X$, sometimes called *factivity* because it asserts that justifications only serve to justify facts. Semantically, one requires that accessibility be reflexive, as usual.

For JK4, LP, and JS5, an additional operation is needed. It is written !, and is called *proof checker* or *positive justification checker*. Axiomatically one adds the scheme $t{:}X \supset !t{:}t{:}X$. Semantically some special requirements are needed. The evidence function must satisfy the condition $\mathcal{E}(t, X) \subseteq \mathcal{E}(!t, t{:}X)$. Also, evidence functions are required to be monotonic: if $\Gamma \in \mathcal{E}(t, X)$ and $\Gamma \mathcal{R} \Delta$, then $\Delta \in \mathcal{E}(t, X)$. And finally, accessibility is required to be transitive, as might be expected. If ! is present, the Iterated Axiom Necessitation Rule can be replaced by a simpler version, but we do not go into details here.

Finally, for JS5 still one more operation is needed, ?, a *negative justification checker*. Axiomatically one adds $\neg t{:}X \supset ?t{:}\neg t{:}X$. We omit the semantic conditions.

For all these logics, soundness and completeness theorems are provable, as are Internalization Theorems and, most importantly, Realization Theorems. Thus for each of the most common modal logics there exists a justification version that, in a straightforward sense, internalizes its proof structure.

The modal logics just mentioned are not all that have been considered. There are multi-modal versions, thought of as logics of *explicit* knowledge. There are versions with both explicit justifications and implicit ones, that is, conventional modal operators. There are versions in which one quantifies over justifications. And there is current work on a first-order version of justification logics with quantifiers over a domain in the usual way. All this is very much in progress.

3 Hybrid Logic

Hybrid logics internalize aspects of their semantics. Possible worlds are represented directly, while accessibility only a little less directly. Probably the best references are [1, 5]. The presentation here is rather minimal, since hybrid logics are generally more familiar than justification logics.

The language is an extension of that of conventional modal logic. One adds a new family of propositional letters, called *nominals*—we use i, j, k, ... for this purpose. Then various additions can be made to this basis. We will only be interested in one that also adds a prefix operator, $@_i$, for each nominal i, so that $@_i\varphi$ is a formula if φ is.

Semantically, nominals are interpreted as being true at exactly one world, and so a nominal can be thought of as a name for a possible world. Then $@_i\varphi$ is taken to be true at an arbitrary possible world if φ is true at the world named by i. The semantics then divides in two, depending on whether one requires that all possible worlds have nominal names, or not. If one requires every world to have a name, we have *named models*, and hybrid logic corresponding to these models is quite rich and elegant. Without such a naming requirement, we have *basic hybrid logic*, still of interest but less so. Unfortunately, I have only been able to work out a combined hybrid/justification logic corresponding to basic hybrid logic—enhancing it to named models is more than I know how to do. I'll say more about this below. The point for now is: in the semantics considered here, not every possible world need have a nominal naming it.

Besides propositional letters, nominals, and @ operators, the language has the usual connectives, $\wedge$, $\vee$, $\neg$, $\supset$, and modal operators. The possibility operator $\Diamond$ is often taken as basic, with $\Box$ characterized in terms of it. And there is an operator $@_i$ for each nominal i. Formulas are built up in the way usual with propositional modal logic, with the additional constructor: if X is a formula, so is $@_iX$ for each nominal i.

For an axiomatization of basic hybrid logic, we have the usual classical and modal axiom schemes and rules for K, together with the following. In these, i, j, ... are any nominals and X, Y, ... are any formulas. This is taken from [5].

$K_@$	$@_i(X \supset Y) \supset (@_iX \supset @_iY)$
self-dual	$@_iX \equiv \neg @_i \neg X$
introduction	$(i \wedge X) \supset @_iX$
ref	$@_ii$
sym	$@_ij \equiv @_ji$
nom	$(@_ij \wedge @_jX) \supset @_iX$
agree	$@_j@_iX \equiv @_iX$
back	$\Diamond @_iX \supset @_iX$

Also in addition to the usual rule of necessitation, there is a similar rule for $@_i$.

$$@\text{-}Necessitation \quad \frac{X}{@_iX}$$

Easily provable are the following.

elimination	$(i \wedge @_i X) \supset X$
bridge	$(\Diamond i \wedge @_i X) \supset \Diamond X$
dual-back	$@_i X \supset \Box @_i X$

We note that the *dual-back* scheme can be used in place of *back*.

Soundness and completeness arguments can be found in [1, 5].

4 Hybrid Justification Logic

We now present a logic combining aspects of basic hybrid logic and justification logic. Rather than give a hybrid version of the simplest justification logic, J, we give one for JT, because the formulation is a little simpler. We call the logic *hybrid*-JT. This is intended to be a first pass at combining nominals and justifications—it is to be hoped that natural formulations of other justification logics and hybrid logics will be found.

4.1 Language and Axioms

We use a combination of the language machinery presented above, together with some additional items. On the hybrid side we have nominals and @ operators, as in Section 3. For building justification terms we have justification constants and variables, and operations + and ·, as in Section 2.1. In addition we have the following. For each nominal i we have a justification term f_i, which we call a *remote fact checker*. For each nominal i we have prefix operators $!_i$ (remote positive justification checker) and $?_i$ (remote negative justification checker). (Reasons for the names will become clear shortly.)

For axioms, of course we have classical tautologies. Then, on the justification side we have the + Axioms and the · Axioms from Section 2.2. In addition, since we are considering a hybrid version of JT, we also assume the Factivity Axiom, $t{:}X \supset X$, from Section 2.5. On the hybrid side we assume all the axioms from Section 3 that do not involve a modal operator, namely $K_@$, *self-dual*, *introduction*, *ref*, *sym*, *nom*, and *agree*. (*back* is missing.)

For rules, we have the usual *modus ponens*. Also we have necessitation for $@_i$ (but not for $\Box$, which is not in the language). The role of justification constants is expanded somewhat. As things were in Section 2.1, constants served as justifications for axioms, which were not further analyzed. But axioms are axioms no matter where we find them. Hence we adopt the following rule, in addition to *Iterated Axiom Necessitation* which we also assume. This rule begins the 'crossover' part, connecting justifications and nominals.

Iterated Remote Axiom Necessitation Let i be any nominal. If X is an axiom and $c_1, c_2, \ldots, c_n$ are constants, then $c_1{:}c_2{:}\ldots c_n{:}@_i X$.

Finally we have axioms that involve both justification and hybrid machinery. Some of the axioms that follow are, in fact, explicit axioms and not schemes. This contrasts with the axioms mentioned above, which are specified via axiom schemes.

Informally, a state that is named by a nominal is a collection of 'facts.' As Wittgenstein has it, "The facts in logical space are the world" [12]. Facts that make up a state should be verifiable as doing so. Consequently we have the following axioms, in which i is an arbitrary nominal and P is an arbitrary *propositional letter*.

remote fact checker $@_i P \supset f_i{:}@_i P$
$@_i \neg P \supset f_i{:}@_i \neg P$

The remaining axioms are given as schemes. For these the idea is that if it is claimed that t serves to justify some formula X at a given state i, it can be checked anywhere whether the claim is correct or not. Then, for any formula X, any justification term t, and any nominal i, the following are axioms.

remote positive justification checker $@_i t{:}X \supset (!_i t){:}@_i t{:}X$
remote negative justification checker $@_i \neg t{:}X \supset (?_i t){:}@_i \neg t{:}X$

This completes the axiom system for *hybrid*-JT. *Constant specifications* are as they were in Section 2.2, except that now in addition to containing formulas of the form $c_1{:}c_2{:}\ldots c_n{:}X$, they also may contain formulas of the form $c_1{:}c_2{:}\ldots c_n{:}@_i X$, where i is a nominal. Now we say a proof *meets constant specification* $\mathcal{C}$ provided that whenever $c_1{:}c_2{:}\ldots c_n{:}X$ is introduced using the *Iterated Axiom Necessitation* rule, then this formula is a member of $\mathcal{C}$, and whenever $c_1{:}c_2{:}\ldots c_n{:}@_i X$ is introduced using the *Remote Axiom Necessitation* rule, this formula is in $\mathcal{C}$. We still use the terminology *axiomatically appropriate* for certain constant specifications—those in which each axiom has specified constants to justify it, both directly and remotely.

4.2 Internalization

We begin with a few preliminary results, then we prove an Internalization Theorem for *hybrid*-JT. Throughout we assume we are using an axiomatically appropriate constant specification. Recall that $\neg X$ abbreviates $X \supset \bot$. The following Lemma is almost immediate.

Lemma 4.1 *For every atomic formula X, and for every nominal i, there are closed justification terms t and u such that both $@_i X \supset t{:}@_i X$ and $@_i \neg X \supset u{:}@_i \neg X$ are provable.*

Proof If X is a propositional letter P, the remote fact checker axioms take care of things. But also, X could be $\bot$. It is easy to show that $@_i \bot \supset \bot$ is a theorem of hybrid logics generally, and so also of *hybrid*-JT. It follows that $@_i \bot \supset t{:}@_i \bot$ is a theorem for every justification term t. Finally, since $\neg\bot$ is $\bot \supset \bot$, which is a tautology, the *Remote Axiom Necessitation* rule gives us $c{:}@_i \neg\bot$, and hence $@_i \neg\bot \supset c{:}@_i \neg\bot$ is trivially provable. ■

The next item plays an important role in the proof of Internalization, and is of interest for its own sake.

Proposition 4.2 *For every formula X, not necessarily atomic, and for every nominal i, there are closed justification terms t and u such that both $@_i X \supset t{:}@_i X$ and $@_i \neg X \supset u{:}@_i \neg X$ are provable in* hybrid-*JT.*

Proof By induction on the degree of X, defined as the number of connectives, @ symbols, and $t{:}$ occurrences in X. The atomic case is covered by Lemma 4.1. Beyond the atomic level we have several cases, and subcases. In some of these we make use of the following facts about hybrid logics in general. Each operator $@_i$ is a normal modal operator, because of the $K_@$ axiom and the $@_i$ rule of necessitation. Consequently $@_i(Y \wedge Z) \equiv (@_i Y \wedge @_i Z)$ is provable in the usual way. Then, using the *self-dual* scheme it follows that $@_i(Y \vee Z) \equiv (@_i Y \vee @_i Z)$ is also provable.

Now suppose X is not atomic, and the result is known for formulas of lower degree.

Case: X is $Y \supset Z$, Positive subcase. We have:

$$\begin{aligned} @_i(Y \supset Z) &\supset @_i(\neg Y \vee Z) \\ &\supset (@_i\neg Y \vee @_i Z) \end{aligned}$$

By the induction hypothesis there are closed justification terms u and v such that both $@_i\neg Y \supset u{:}@_i\neg Y$ and $@_i Z \supset v{:}@_i Z$ are provable, and hence so is the following.

$$@_i(Y \supset Z) \supset (u{:}@_i\neg Y \vee v{:}@_i Z)$$

The formula $\neg Y \supset (Y \supset Z)$ is a tautology, so by *iterated remote axiom necessitation* we have $c{:}@(\neg Y \supset (Y \supset Z))$, for some constant c. Also $@_i(\neg Y \supset (Y \supset Z)) \supset (@_i\neg Y \supset @_i(Y \supset Z))$ is a $K_@$ axiom and hence by *iterated axiom necessitation* so is $d{:}[@_i(\neg Y \supset (Y \supset Z)) \supset (@_i\neg Y \supset @_i(Y \supset Z))]$ for some constant d. Then using the $\cdot$ axiom, we derive $(d\cdot c){:}(@_i\neg Y \supset @_i(Y \supset Z))$. Then by the $\cdot$ axiom again, we have $u{:}@_i\neg Y \supset ((d \cdot c) \cdot u){:}@_i(Y \supset Z)$.

By an argument similar to the one given in the previous paragraph we can prove the formula $v{:}@_Z \supset ((f\cdot e)\cdot v){:}@_i(Y \supset Z)$, where e is a constant remotely justifying the axiom $@_i(Z \supset (Y \supset Z))$ and f is a constant justifying the $K_@$ axiom $@_i(Z \supset (Y \supset Z)) \supset (@_i Z \supset @_i(Y \supset Z))$. Then, using the $+$ axioms, we have the following.

$$\begin{aligned} @_i(Y \supset Z) &\supset (u{:}@_i\neg Y \vee v{:}@_i Z) \\ &\supset [((d \cdot c) \cdot u){:}@_i(Y \supset Z) \vee ((f \cdot e) \cdot v){:}@_i(Y \supset Z)] \\ &\supset [((d \cdot c) \cdot u + (f \cdot e) \cdot v){:}@_i(Y \supset Z) \vee ((d \cdot c) \cdot u + (f \cdot e) \cdot v){:}@_i(Y \supset Z)] \\ &\supset ((d \cdot c) \cdot u + (f \cdot e) \cdot v){:}@_i(Y \supset Z) \end{aligned}$$

Case: X is $Y \supset Z$, Negative subcase. This time, using the induction hypothesis, there are closed justification terms u and v so that the following is provable.

$$\begin{aligned} @_i\neg(Y \supset Z) &\supset @_i(Y \wedge \neg Z) \\ &\supset (@_i Y \wedge @_i\neg Z) \\ &\supset (u{:}@_i Y \wedge v{:}@_i\neg Z) \end{aligned}$$

Using *iterated remote axiom necessitation* we have $c{:}@_i[Y \supset (\neg Z \supset \neg(Y \supset Z))]$ for some constant c. And since instances of axiom $K_@$ are involved, there are justification constants d and e such that $d{:}\{@_i[Y \supset (\neg Z \supset \neg(Y \supset Z))] \supset [@_i Y \supset @_i(\neg Z \supset \neg(Y \supset Z))]\}$ and $e{:}[@_i(\neg Z \supset \neg(Y \supset Z)) \supset (@_i\neg Z \supset @_i\neg(Y \supset Z))]$ are provable. Then making use of the $\cdot$ axiom, we have $(d \cdot c){:}[@_i Y \supset @_i(\neg Z \supset \neg(Y \supset Z))]$, and hence again $u{:}@_i Y \supset ((d \cdot c) \cdot u){:}@_i(\neg Z \supset \neg(Y \supset Z))$. At this point we have

$$@_i\neg(Y \supset Z) \supset [((d \cdot c) \cdot u){:}@_i(\neg Z \supset \neg(Y \supset Z)) \wedge v{:}@_i\neg Z]$$

Next, in a similar way, we get

$$@_i\neg(Y \supset Z) \supset [(e \cdot ((d \cdot c) \cdot u)){:}(@_i\neg Z \supset @_i\neg(Y \supset Z)) \wedge v{:}@_i\neg Z]$$

and finally we have the following.

$$@_i\neg(Y \supset Z) \supset ((e \cdot ((d \cdot c) \cdot u)) \cdot v){:}@_i\neg(Y \supset Z)$$

Case: X is $@_j Y$, Positive subcase. Using the induction hypothesis we have the provability of $@_j Y \supset u{:}@_j Y$ for some u. Then using *@-necessitation* and the $K_@$ axiom, we also have $@_i @_j Y \supset @_i u{:}@_j Y$. Applying the *remote positive justification checker* axiom we get $@_i @_j Y \supset (!_i u){:}@_i u{:}@_j Y$.

The *factivity axiom* gives us $u{:}@_j Y \supset @_j Y$. Then *iterated remote axiom necessitation* gives $c{:}@_i(u{:}@_j Y \supset @_j Y)$ for some constant c. Since $@_i(u{:}@_j Y \supset @_j Y) \supset (@_i u{:}@_j Y \supset @_i @_j Y)$ is a $K_@$ axiom, there is a constant d that justifies it. Then, using the $\cdot$ axiom, we have provability of $(d \cdot c){:}(@_i u{:}@_j Y \supset @_i @_j Y)$, and hence also of $(!_i u){:}@_i u{:}@_j Y \supset ((d \cdot c) \cdot !_i u){:}@_i @_j Y$. Combining things, we have shown provability of $@_i @_j Y \supset ((d \cdot c) \cdot !_i u){:}@_i @_j Y$.

Case: X is $@_j Y$, Negative subcase. This is similar to the preceding case, but $?_i$ must be used instead of $!_i$. We omit details.

Case: X is $w{:}Y$, Positive subcase. Covered by the *remote positive justification checker* axiom.

Case: X is $w{:}Y$, Negative subcase. Covered by the *remote negative justification checker* axiom.

■

We now arrive at the Internalization Theorem itself. As noted earlier, it originates with Artemov [2], and says that Justification Logics are capable of representing their own proof structures. Here is a version for *hybrid*-JT. The proof of it is as significant as the result itself—it amounts to showing that axiomatic proofs internalize as justification terms. What follows is a modification of the original Artemov proof.

Theorem 4.3 (Internalization) *If X is a theorem of* hybrid-JT*, then there is a closed justification term t such that $t{:}X$ is also a theorem.*

Proof Let $X_1, X_2, \ldots, X_n = X$ be a proof of X in the *hybrid*-JT axiom system of section 4.1. We show that for each $k \le n$ there is some closed justification term t_k such that $t_k{:}X_k$ is provable. The case $k = n$ establishes the Theorem. The proof is by induction on k (although the induction hypothesis is actually needed for only one case).

Assume the result is known for all $m < k$; we show it holds for k as well. There are several cases to consider.

Case: Axiom Suppose X_k is an axiom. Then $c{:}X_k$ is provable for some justification constant c, by *Iterated Axiom Necessitation.*

Case: Modus Ponens Suppose X_a and $X_b = (X_a \supset X_k)$ are formulas with $a, b < k$. By the induction hypothesis there are closed terms t_a and t_b so that $t_a{:}X_a$ and $t_b{:}(X_a \supset X_k)$ are both provable. Then, using the $\cdot$ axiom, and *modus ponens*, $(t_b \cdot t_a){:}X_k$ will also be provable.

Case: @-Necessitation Suppose $X_k = @_i X_m$ for some $m < k$. Since $@_i X_m$ is provable, by Proposition 4.2 there is some closed term t such that $t{:}@_i X_m$ is provable.

Case: Iterated Axiom Necessitation Suppose $X_k = c_1{:}c_2{:}\ldots c_n{:}Z$ where Z is an axiom. Then $c_0{:}c_1{:}c_2{:}\ldots c_n{:}Z$ also follows by *iterated axiom necessitation*, and this is $c_0{:}X_k$.

Case: Iterated Remote Axiom Necessitation Similar to the previous case.

■

4.3 Semantics

The semantics given earlier can be easily modified to one appropriate for the mixed system we have been considering. A *hybrid*-JT model is a structure $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$ that meets the conditions for being a J model, from section 2.3, with certain additional conditions imposed, which we now describe.

First, since we have been considering a hybrid version of JT, and not of J, the accessibility relation $\mathcal{R}$ must be reflexive.

Next, the valuation $\mathcal{V}$ is modified in the usual hybrid way so that for each nominal i, the set $\mathcal{V}(i)$ is a singleton. If $\mathcal{V}(i) = \{\Gamma\}$, in an abuse of notation we will generally write $\mathcal{V}(i) = \Gamma$.

Since we have three additional operators on justification terms, the definition of $\mathcal{E}$, the admissible evidence function, must be extended.

f_i**, remote fact checker** For a propositional letter P and a nominal i, if $\mathcal{V}(i) \in \mathcal{V}(P)$ then $\mathcal{E}(f_i, @_i P) = \mathcal{G}$, and if $\mathcal{V}(i) \notin \mathcal{V}(P)$ then $\mathcal{E}(f_i, @_i P) = \emptyset$.

$!_i$**, remote positive justification checker** If $\mathcal{V}(i) \in \mathcal{E}(t, X)$ then $\mathcal{E}(!_i t, @_i t{:}X) = \mathcal{G}$.

$?_i$ **remote negative justification checker** If $\mathcal{V}(i) \notin \mathcal{E}(t, X)$ then $\mathcal{E}(?_i t, @_i \neg t{:}X) = \mathcal{G}$.

Finally since we have @ in the language, the definition of $\Vdash$ must be extended.

$$\mathcal{M}, \Gamma \Vdash @_i X \iff \mathcal{M}, \mathcal{V}(i) \Vdash X$$

This completes the specification of models for *hybrid*-JT.

4.4 Completeness

Let $\mathcal{C}$ be an axiomatically appropriate constant specification, fixed for this section. We will show that a formula X is provable in the *hybrid*-JT axiom system of section 4.1, using $\mathcal{C}$, if and only if X is true at all states of all *hybrid*-JT models of section 4.3 that meet $\mathcal{C}$ if and only if X is true at all states of all models that meet $\mathcal{C}$ and are fully explanatory.

Actually, the soundness direction of the result just stated is straightforward, and is left to the reader. For the completeness direction we use a direct combination of the justification logic argument from [8] and the hybrid argument from [5]. We sketch the ideas, and refer to these sources for more details.

First, from [5], is the following result (reproduced with original theorem numbering from that reference). Note that K_h-MCS means a maximally consistent set in a particular hybrid logic that corresponds to the one we are using. Also a nominal i is said to be a *name for* a K_h-MCS set Γ if i is in Γ, and Γ is *named* if there is a name for it.

Lemma 7.24 *Let Γ be a K_h-MCS. For every nominal i, let Δ_i be $\{\phi \mid @_i\phi \in \Gamma\}$. Then:*

i For every nominal i, Δ_i is a K_h-MCS that contains i.

ii For all nominals i and j, if $i \in \Delta_j$, then $\Delta_j = \Delta_i$.

iii For all nominals i and j, $@_i\phi \in \Delta_j$ iff $@_i\phi \in \Gamma$.

iv If k is a name for Γ, then $\Gamma = \Delta_k$.

The proof of this Lemma, given in [5], only uses machinery that is also available in our *hybrid*-JT, and so the results may be applied here as well. Now here is a sketch of our completeness argument.

Call a set S of formulas in the *hybrid*JT language *inconsistent* if there is some finite subset $\{s_1, \ldots, s_n\} \subseteq S$ such that $(s_1 \wedge \ldots \wedge x_n) \supset \bot$ is provable. Call S *consistent* if S is not inconsistent. Recall we are assuming a fixed axiomatically appropriate constant specification $\mathcal{C}$ is used.

Let S be a consistent set of formulas. We construct a model in which S is satisfied. The construction is in two stages.

Stage 1. For the first stage of the construction, let $\mathcal{G}_1$ be the collection of all maximally consistent sets of formulas. For each $\Gamma \in \mathcal{G}_1$, set $\Gamma^\sharp$ to be $\{X \mid t{:}X \in \Gamma\}$ for some justification term t. Let $\Gamma \mathcal{R}_1 \Delta$ if $\Gamma^\sharp \subseteq \Delta$. For each propositional letter P (including nominals), set $\mathcal{V}_1(P) = \{\Gamma \in \mathcal{G}_1 \mid P \in \Gamma\}$. And finally, set $\mathcal{E}_1(t, X) = \{\Gamma \in \mathcal{G}_1 \mid t{:}X \in \Gamma\}$. We now have a structure $\mathcal{M}_1 = \langle \mathcal{G}_1, \mathcal{R}_1, \mathcal{E}_1, \mathcal{V}_1 \rangle$.

The structure $\mathcal{M}_1$ meets some, but not all, of the conditions for being a *hybrid*-JT model. Because of the *factivity* axiom, $\mathcal{R}_1$ is reflexive. The admissible evidence function meets the conditions given in section 2.3. But for a nominal i, $\mathcal{V}(i)$ might not be a singleton.

Stage 2. We have the consistent set S, extend it to a maximally consistent set Σ. Of course $\Sigma \in \mathcal{G}_1$. For a nominal i, let $\Delta_i = \{Z \mid @_i Z \in \Sigma\}$. By Lemma 7.24 part *i*, every Δ_i is maximally consistent, and is named by i, and so $\Delta_i \in \mathcal{G}_1$. Now let $\mathcal{G}_2$ be the subset of $\mathcal{G}_1$ that is generated by Σ together with the various Δ_i. That is, each member of $\mathcal{G}_2$ is the last term of a sequence in which every term (except the last) is in the relation $\mathcal{R}_1$ to the next, and in which the first term is either Σ or some Δ_i. Let $\mathcal{R}_2$, $\mathcal{E}_2$, and $\mathcal{V}_2$ be the restrictions of $\mathcal{R}_1$, $\mathcal{E}_1$, and $\mathcal{V}_1$ to $\mathcal{G}_2$. We have the structure $\mathcal{M}_2 = \langle \mathcal{G}_2, \mathcal{R}_2, \mathcal{E}_2, \mathcal{V}_2 \rangle$, and this is actually the model we want, though proving it takes a certain amount of work. Note, incidentally, that if $\Gamma \in \mathcal{G}_2$, $\Delta \in \mathcal{G}_1$, and $\Gamma \mathcal{R}_1 \Delta$ then $\Delta \in \mathcal{G}_2$. This plays a role below.

We show that if any member of $\mathcal{G}_2$ contains $@_i X$, they all do, for any formula X. The property in question holds for the subset of $\mathcal{G}_2$ consisting of Σ and the various Δ_j, by Lemma 7.24, part *iii*. Since $\mathcal{G}_2$ is the subset of $\mathcal{G}_1$ generated by these members, it is enough to show that, for any $\Gamma, \Delta \in \mathcal{G}_2$, if $\Gamma \mathcal{R}_2 \Delta$ then $@_i X \in \Gamma$ if and only if $@_i X \in \Delta$. The argument has two parts. Suppose first that $@_i X \in \Gamma$. By Proposition 4.2, and the maximal consistency of Γ, $t{:}@_i X \in \Gamma$ for some justification term t, so $@_i X \in \Delta$, by definition of $\mathcal{R}_2$ (actually, of $\mathcal{R}_1$). Next, suppose that $@_i X \notin \Gamma$. By maximal consistency, $\neg @_i X \in \Gamma$, and by *self-dual*, $@_i \neg X \in \Gamma$. Then by an argument similar to the previous one, $@_i \neg X \in \Delta$ and then again, $@_i X \notin \Delta$.

Next we show that for any nominal i and any $\Omega \in \mathcal{G}_2$, if $i \in \Omega$ then $\Omega = \Delta_i$. Well, suppose $i \in \Omega$. We show $\Delta_i \subseteq \Omega$. Let $X \in \Delta_i$. Then $@_i X \in \Sigma$, so $@_i X \in \Omega$, by the previous paragraph. Since $i \in \Omega$, $X \in \Omega$, by *elimination*. The reverse inclusion is similar, making use of maximal consistency and negation.

It follows from what was just shown that, for each nominal i, $\mathcal{V}_2(i)$ is a singleton, in fact, $\mathcal{V}_2(i) = \Delta_i$. We also easily get: for any $\Gamma \in \mathcal{G}_2$, $@_i X \in \Gamma$ if and only if $X \in \Delta_i$.

Now it is easy to verify that $\mathcal{M}_2 = \langle \mathcal{G}_2, \mathcal{R}_2, \mathcal{E}_2, \mathcal{V}_2 \rangle$ has all the appropriate properties. We verify just a few items.

Suppose $\mathcal{V}_2(i) \notin \mathcal{E}_2(t, X)$. We show $\mathcal{E}_2(?_i t, @_i \neg t{:}X) = \mathcal{G}_2$, the remote negative justification checker condition for $\mathcal{E}_2$. Well, $\mathcal{V}_2(i) = \Delta_i$, so by definition of $\mathcal{E}_2$ (and $\mathcal{E}_1$), $t{:}X \notin \Delta_i$, and by maximal consistency, $\neg t{:}X \in \Delta_i$. Then $@_i \neg t{:}X \in \Sigma$, and hence $@_i \neg t{:}X$ is in every member of $\mathcal{G}_2$. It follows, using the *remote negative justification checker* axiom, that $(?_i t){:}@_i \neg t{:}X$ is in every member of $\mathcal{G}_2$, and so $\mathcal{E}_2(?_i t, @_i t{:}\neg X) = \mathcal{G}_2$.

Next we establish the familiar

Truth Lemma For any $\Gamma \in \mathcal{G}_2$ and any formula X,

$$\mathcal{M}_2, \Gamma \Vdash X \iff X \in \Gamma$$

The proof is by induction, as usual. Propositional cases are standard, and are omitted. For the @ case, we proceed as follows. $\mathcal{M}_2, \Gamma \Vdash @_i X$ is equivalent to $\mathcal{M}_2, \Delta_i \Vdash X$, using the fact that $\mathcal{V}_2(i) = \Delta_i$. By the induction hypothesis, this is equivalent to $X \in \Delta_i$. We showed above that this is equivalent to $@_i X \in \Gamma$ for (every) Γ.

Finally we check the justification term case. The two directions require different arguments.

Suppose first that $t{:}X \in \Gamma$. Then $\Gamma \in \mathcal{E}_2(t, X)$, by definition. Also, for every $\Delta \in \mathcal{G}_2$ with $\Gamma \mathcal{R}_2 \Delta$, we have $X \in \Delta$ by definition of accessibility. By the induction hypothesis, $\mathcal{M}_2, \Delta \Vdash X$ for all accessible Δ. It follows that $\mathcal{M}_2, \Gamma \Vdash t{:}X$. In the other direction, if $t{:}X \notin \Gamma$, then $\Gamma \notin \mathcal{E}_2(t, X)$, hence $\mathcal{M}_2, \Gamma \nVdash t{:}X$.

We have established that $\mathcal{M}_2$ is a model, and of course the consistent set S is satisfied in it, at Σ. It remains to show that $\mathcal{M}_2 = \langle \mathcal{G}_2, \mathcal{R}_2, \mathcal{V}_2, \mathcal{E}_2 \rangle$ satisfies the fully explanatory condition, from Section 2.3. Suppose $\mathcal{M}_2, \Delta \Vdash X$ for every $\Delta \in \mathcal{G}_2$ such that $\Gamma \mathcal{R}_2 \Delta$. We will show that $\mathcal{M}_2, \Gamma \Vdash t{:}X$ for some justification term t. Stating this in contrapositive form, and making use of the Truth Lemma, we show that if $t{:}X \notin \Gamma$ for every justification term t, then $X \notin \Delta$ for some Δ such that $\Gamma \mathcal{R}_2 \Delta$. So, suppose that for every justification term t, $t{:}X \notin \Gamma$.

We begin by showing that $\Gamma^\sharp \cup \{\neg X\}$ is consistent. If $\Gamma^\sharp \cup \{\neg X\}$ were inconsistent, for some $G_1, \ldots, G_n \in \Gamma^\sharp$, $(G_1 \wedge G_2 \wedge \ldots \wedge G_n \wedge \neg X) \supset \bot$ would be provable, and hence so would $(G_1 \supset (G_2 \supset \ldots \supset (G_n \supset X) \ldots))$. Using Internalization, Proposition 4.2, there is a justification term u such that $u{:}(G_1 \supset (G_2 \supset \ldots \supset (G_n \supset X) \ldots))$ is provable. For each $i = 1, \ldots, n$, since $G_i \in \Gamma^\sharp$, there is some justification term s_i such that $s_i{:}Y_i \in \Gamma$. Now by repeated use of the $\cdot$-axiom, we conclude $(\ldots((u \cdot s_1) \cdot s_2) \cdot \ldots \cdot s_n){:}X \in \Gamma$, contradicting the assumption that $t{:}X \notin \Gamma$ for every t.

Now that we know $\Gamma^\sharp \cup \{\neg X\}$ is consistent, we can extend it to a maximal consistent set Δ. By definition, $\Delta \in \mathcal{G}_1$ and $\Gamma \mathcal{R}_1 \Delta$. Since $\Gamma \in \mathcal{G}_2$, then also $\Delta \in \mathcal{G}_2$ and $\Gamma \mathcal{R}_2 \Delta$. And of course $X \notin \Delta$.

Definition 4.4 For a consistent set S, we call the model $\mathcal{M}_2$ constructed above the *S-canonical model*.

Completeness now follows in the standard way. If X is not a theorem of *hybrid*-JT, then $\{\neg X\}$ is consistent, hence satisfied in the $\{\neg X\}$ canonical model, and so X is falsified at a possible world of a *hybrid*-JT model meeting the fully explanatory condition.

5 The Realization Theorem

If we think of the $\Box$ of a modal logic as an implicit knowledge operator, $\Box X$ asserts that we know X. But in a justification logic, $t{:}X$ asserts we know X for an explicit reason, t. In this sense justification logics provide an analysis of corresponding modal logics. A formal statement of this is embodied in the *Realization Theorem*, which was first proved by Artemov in [3], in a version that connected the justification logic LP with the modal logic S4. Since then Realization Theorems have been provided for other logics, and additional ways of proving it have been developed. In this section we will state and prove a Realization Theorem connecting *hybrid*-T with its justification version *hybrid*-JT.

First we consider the easy direction. Suppose, in any formula of *hybrid*-JT, we replace every justification term with an occurrence of $\Box$. So, for instance, $t{:}(u{:}X \supset X)$ becomes $\Box(\Box X \supset X)$. Call this the *forgetful functor*. It is easy to check that if we apply the forgetful functor to any axiom of *hybrid*-JT, the result is an axiom of *hybrid*-T. Likewise the forgetful functor turns rules of derivation for *hybrid*-JT into rules of *hybrid*-T. It follows that the forgetful functor converts entire proofs using justifications into their counterpart modal proofs. Thus we have the following.

Proposition 5.1 *If X^j is a theorem of* hybrid-*JT, with respect to any constant specification, and X is the result of applying the forgetful functor to X^j, then X is a theorem of* hybrid-*T.*

As we said, this is the easy direction. In fact the forgetful functor not only maps *hybrid*-JT to *hybrid*-T, but the mapping is onto. This is the hard part, and is an immediate consequence of the following.

Theorem 5.2 (Realization) *Let X be a theorem of* hybrid-*T. There is some formula X^j of* hybrid-*JT such that:*

1. *in X^j negative occurrences of $\Box$ in X have been replaced with distinct justification variables;*
2. *in X^j positive occurrences of $\Box$ in X have been replaced with justification terms, not necessarily variables;*
3. *X^j is provable in* hybrid-*JT using some injective constant specification.*

In the theorem above, X^j is called a *realization* of X. Applying the forgetful functior to X^j yields X and so we have the onto result mentioned earlier, but there is more to say. Notice that negative $\Box$ occurrences in X become justification variables. This says that theorems of *hybrid*-T have a hidden input/output structure—positive occurrences of $\Box$ implicitly represent reasons that can be computed from the reasons implicitly represented by negative occurrences of $\Box$.

In the remainder of this section we will prove the Realization Theorem. Artemov's original proof was constructive, making use of modal sequent calculus proofs. Since then other constructive proofs have been developed, and there is a non-constructive one from [8]. It is this non-constructive argument on which we base the present proof.

5.1 Realization, Weakly

What is established in this section is not the 'real' Realization Theorem. We formulate an embedding result, but it does not have the simple form we want. That will come in the next section, making use of the version shown here. In [8] the weaker version was considered more carefully and became a partial analysis of the role of + in justification logics. We do not take this care now, and the work of this section should be thought of as simply a preliminary to the main event.

Let φ be a formula in the language of *hybrid*-T. We want to construct a family of candidates for *weak realizations* for φ. To do this, we must associate such a family with every subformula of φ as well, and positive and negative subformulas of the form $\Box X$ are treated in different ways. In negative subformulas of this form, the $\Box$ operator must be replaced with justification variables, and this replacement must meet the restrction that different negative occurrences of $\Box$ are replaced with distinct justification variables. In order to manage this as simply as possible we introduce a small amount of machinery.

Let us assume that, in φ, each *negative* occurrence of $\Box$ is numbered, 1, 2, ..., with numbering going from left to right. (This ordering is just to be specific—any ordering would do.) We call these numbers *indexes* of negative occurrences of $\Box$, and we say φ is *indexed*. Sometimes we indicate indexes as subscripts, though they are not an official part of the language. So, for instance, if we displayed the indexes in $\Box X \supset (\Box Y \supset \Box X)$ we would see $\Box_1 X \supset (\Box_2 Y \supset \Box X)$. Note that positive occurrences do not have indexes.

We set aside a family, x_1, x_2, ..., of distinct justification variables. When we replace a negative occurrence of $\Box$ with a justification variable, if the occurrence has index i, we will use justification variable x_i. Thus $\Box_i$ in φ turns into x_i, this choice is made ahead of time, and is fixed.

Now we say how we associate a family of justification formulas with each subformula of φ, which we assume is indexed. If X is a subformula, the set associated with it is denoted $\langle\!\langle X\rangle\!\rangle_\varphi$.

Definition 5.3 Let φ be a formula of *hybrid*-T, indexed. For each subformula Z of φ we define a set $\langle\!\langle Z\rangle\!\rangle_\varphi$ of formulas of *hybrid*-JT. We have the following cases.

1. A is an atomic subformula of φ (including $\bot$ and nominals). Let $\langle\!\langle A\rangle\!\rangle_\varphi = \{A\}$.
2. $X \supset Y$ is a subformula of φ. Let $\langle\!\langle X \supset Y\rangle\!\rangle_\varphi = \{A \supset B \mid A \in \langle\!\langle X\rangle\!\rangle_\varphi \text{ and } B \in \langle\!\langle Y\rangle\!\rangle_\varphi\}$.
3. $@_i X$ is a subformula of φ. Let $\langle\!\langle @_i X\rangle\!\rangle_\varphi = \{@_i A \mid A \in \langle\!\langle X\rangle\!\rangle_\varphi\}$.
4. $\Box X$ is a negative subformula of φ with index n, so that $\Box X = \Box_n X$.
   $\langle\!\langle \Box X\rangle\!\rangle_\varphi = \{x_n{:}A \mid A \in \langle\!\langle X\rangle\!\rangle_\varphi\}$.
5. $\Box X$ is a positive subformula of φ.
   $\langle\!\langle \Box X\rangle\!\rangle_\varphi = \{t{:}(A_1 \vee \ldots \vee A_n) \mid A_1, \ldots, A_n \in \langle\!\langle X\rangle\!\rangle_\varphi,\ t \text{ is any justification term}\}$

If $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V}\rangle$ is a *hybrid*-JT model, by ignoring the admissible evidence function $\mathcal{E}$ it can also be understood as a *hybrid*-T model. As a *hybrid*-JT model, justification terms t play a role; as a *hybrid*-T model $\Box$ plays a corresponding role. Different formal languages are involved. To keep this dual usage of the model clear, we will write $\mathcal{M}, \Gamma \Vdash_{h\mathsf{JT}} X$ to indicate that the hybrid-justification formula X evaluates to true at world Γ of $\mathcal{M}$, and we will write $\mathcal{M}, \Gamma \Vdash_{h\mathsf{T}} X$ to indicate that the hybrid-modal formula X evaluates to true at Γ. Details should be obvious, and we omit them.

We extend notation a bit. If S is a *set of justification formulas*, we write $\mathcal{M}, \Gamma \Vdash_{h\mathsf{JT}} S$ to mean $\mathcal{M}, \Gamma \Vdash_{h\mathsf{JT}} X$ for every $X \in S$, and we write $\mathcal{M}, \Gamma \nVdash_{h\mathsf{JT}} S$ to mean $\mathcal{M}, \Gamma \nVdash_{h\mathsf{JT}} X$ for every $X \in S$. Notice that if $\mathcal{M}, \Gamma \Vdash_{h\mathsf{JT}} S$ is not true, it does not follow that $\mathcal{M}, \Gamma \nVdash_{h\mathsf{JT}} S$ is, and similarly the other way around (unless S is a singleton).

In Section 4.4, in the course of the completeness argument, we showed how to construct S-canonical *hybrid*-JT models, Definition 4.4. We now show a fundamental result that connects the modal and the justification aspects of these canonical models. Apart from axiomatic adequacy, any constant specification will do for what follows.

Theorem 5.4 *Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V}\rangle$ be a canonical model, and let φ be a formula in the* hybrid-*modal language, assumed indexed. For each $\Gamma \in \mathcal{G}$:*

1. *If Z is a positive subformula of φ and $\mathcal{M}, \Gamma \nVdash_{hJT} \langle\!\langle Z\rangle\!\rangle_\varphi$ then $\mathcal{M}, \Gamma \nVdash_{hT} Z$.*
2. *If Z is a negative subformula of φ and $\mathcal{M}, \Gamma \Vdash_{hJT} \langle\!\langle Z\rangle\!\rangle_\varphi$ then $\mathcal{M}, \Gamma \Vdash_{hT} Z$.*

Proof The proof is by induction on the complexity of Z. The atomic case is simple, so we concentrate on the more complex ones.

Positive Implication Suppose Z is $(X \supset Y)$, Z is a positive subformula of φ, $\mathcal{M}, \Gamma \nVdash_{h\mathsf{JT}} \langle\!\langle X \supset Y\rangle\!\rangle_\varphi$, and the result is known for X (which occurs negatively in φ) and for Y (which occurs positively).

Let $X' \in \langle\!\langle X\rangle\!\rangle_\varphi$, and $Y' \in \langle\!\langle Y\rangle\!\rangle_\varphi$. Then $(X' \supset Y') \in \langle\!\langle X \supset Y\rangle\!\rangle_\varphi$, so $\mathcal{M}, \Gamma \nVdash_{h\mathsf{JT}} (X' \supset Y')$. It follows that $\mathcal{M}, \Gamma \Vdash_{h\mathsf{JT}} X'$ and $\mathcal{M}, \Gamma \nVdash_{h\mathsf{JT}} Y'$. Since X' and Y' were arbitrary, it follows that $\mathcal{M}, \Gamma \Vdash_{h\mathsf{JT}} \langle\!\langle X\rangle\!\rangle_\varphi$ and $\mathcal{M}, \Gamma \nVdash_{h\mathsf{JT}} \langle\!\langle Y\rangle\!\rangle_\varphi$. Then by the induction hypothesis, $\mathcal{M}, \Gamma \Vdash_{h\mathsf{T}} X$ and $\mathcal{M}, \Gamma \nVdash_{h\mathsf{T}} Y$, and hence $\mathcal{M}, \Gamma \nVdash_{h\mathsf{T}} (X \supset Y)$.

Negative Implication Suppose Z is $(X \supset Y)$, Z is a negative subformula of φ, $\mathcal{M}, \Gamma \Vdash_{h\mathsf{JT}} \langle\!\langle X \supset Y \rangle\!\rangle_\varphi$, and the result is known for X (occurring positively) and Y (occurring negatively).

If $\mathcal{M}, \Gamma \nVdash_{h\mathsf{JT}} \langle\!\langle X \rangle\!\rangle_\varphi$, by the induction hypothesis $\mathcal{M}, \Gamma \nVdash_{h\mathsf{T}} X$, and hence $\mathcal{M}, \Gamma \Vdash_{h\mathsf{T}} (X \supset Y)$. Now suppose $\mathcal{M}, \Gamma \nVdash_{h\mathsf{JT}} \langle\!\langle X \rangle\!\rangle_\varphi$ is not the case. Then for some $X' \in \langle\!\langle X \rangle\!\rangle_\varphi$, $\mathcal{M}, \Gamma \Vdash_{h\mathsf{JT}} X'$. Let Y' be an arbitrary member of $\langle\!\langle Y \rangle\!\rangle_\varphi$. Then $(X' \supset Y') \in \langle\!\langle X \supset Y \rangle\!\rangle_\varphi$, so by our assumptions $\mathcal{M}, \Gamma \Vdash_{h\mathsf{JT}} (X' \supset Y')$. Since $\mathcal{M}, \Gamma \Vdash_{h\mathsf{JT}} X'$, then $\mathcal{M}, \Gamma \Vdash_{h\mathsf{JT}} Y'$. Since Y' was arbitrary, we have that $\mathcal{M}, \Gamma \Vdash_{h\mathsf{JT}} \langle\!\langle Y \rangle\!\rangle_\varphi$, so by the induction hypothesis, $\mathcal{M}, \Gamma \Vdash_{h\mathsf{T}} Y$, and again $\mathcal{M}, \Gamma \Vdash_{h\mathsf{T}} (X \supset Y)$.

Positive Necessity Suppose Z is $\Box X$, Z is a positive subformula of φ, $\mathcal{M}, \Gamma \nVdash_{h\mathsf{JT}} \langle\!\langle \Box X \rangle\!\rangle_\varphi$, and the result is known for X (occurring positively in φ). This is the one case that uses the fact that $\mathcal{M}$ is a canonical model.

Let $\Delta_0 = \Gamma^\sharp \cup \{\neg Z \mid Z \in \langle\!\langle X \rangle\!\rangle_\varphi\}$. We first show that Δ_0 is consistent (using any axiomatically adequate constant specification). Well, suppose not. Then for some $Y_1, \ldots, Y_k \in \Gamma^\sharp$ and some $X_1, \ldots, X_n \in \langle\!\langle X \rangle\!\rangle_\varphi$, $(Y_1 \wedge \ldots \wedge Y_k \wedge \neg X_1 \wedge \ldots \wedge \neg X_n) \supset \bot$ is provable, and hence so is $(Y_1 \supset (Y_2 \supset \ldots (Y_k \supset (X_1 \vee \ldots \vee X_n)) \ldots))$. By Theorem 4.3, Internalization, there is a closed justification term u such that

$$u{:}(Y_1 \supset (Y_2 \supset \ldots (Y_k \supset (X_1 \vee \ldots \vee X_n)) \ldots))$$

is provable. Since each $Y_i \in \Gamma^\sharp$, for some justification term t_i, $t_i{:}Y_i \in \Gamma$. Repeated use of the $\cdot$ axiom scheme, and *modus ponens*, yields the provability of

$$(t_1{:}Y_1 \supset (t_2{:}Y_2 \supset \ldots (t_k{:}Y_k \supset (((u \cdot t_1) \cdot t_2) \cdot \ldots \cdot t_k){:}(X_1 \vee \ldots \vee X_n)) \ldots))$$

and since Γ is maximally consistent, $(((u \cdot t_1) \cdot t_2) \cdot \ldots \cdot t_k){:}(X_1 \vee \ldots \vee X_n) \in \Gamma$. It follows by the Truth Lemma that $\mathcal{M}, \Gamma \Vdash_{h\mathsf{JT}} (((u \cdot t_1) \cdot t_2) \cdot \ldots \cdot t_k){:}(X_1 \vee \ldots \vee X_n)$, contradicting the initial assumption that $\mathcal{M}, \Gamma \nVdash_{h\mathsf{JT}} \langle\!\langle \Box X \rangle\!\rangle_\varphi$. Thus Δ_0 is consistent.

The set Δ_0 extends to a maximally consistent set, Δ, and $\Gamma^\sharp \subseteq \Delta$. Since $\Gamma \in \mathcal{G}$, and $\mathcal{G}$ is generated, it follows that $\Delta \in \mathcal{G}$ as well. Since $\{\neg Z \mid Z \in \langle\!\langle X \rangle\!\rangle_\varphi\} \subseteq \Delta$, using the Truth Lemma, $\mathcal{M}, \Delta \nVdash_{h\mathsf{JT}} \langle\!\langle X \rangle\!\rangle_\varphi$. By the induction hypothesis, $\mathcal{M}, \Delta \nVdash_{h\mathsf{T}} X$, and since $\Gamma \mathcal{R} \Delta$, $\mathcal{M}, \Gamma \nVdash_{h\mathsf{T}} \Box X$.

Negative Necessity Suppose Z is $\Box X$, Z is a negative subformula of φ, $\mathcal{M}, \Gamma \Vdash_{h\mathsf{JT}} \langle\!\langle \Box X \rangle\!\rangle_\varphi$, and the result is known for (the negatively occurring) X. Say $\Box X$ has index n in φ, that is, $\Box X$ is $\Box_n X$.

Let $X' \in \langle\!\langle X \rangle\!\rangle_\varphi$. Then $x_n{:}X' \in \langle\!\langle \Box X \rangle\!\rangle_\varphi$, and so $\mathcal{M}, \Gamma \Vdash_{h\mathsf{JT}} x_n{:}X'$. If Δ is any member of $\mathcal{G}$ with $\Gamma \mathcal{R} \Delta$, we must have $\mathcal{M}, \Delta \Vdash_{h\mathsf{JT}} X'$. Since X' was arbitrary, $\mathcal{M}, \Delta \Vdash_{h\mathsf{JT}} \langle\!\langle X \rangle\!\rangle_\varphi$, so by the induction hypothesis, $\mathcal{M}, \Delta \Vdash_{h\mathsf{T}} X$. And since Δ was arbitrary, $\mathcal{M}, \Gamma \Vdash_{h\mathsf{T}} \Box X$.

Positive @ Suppose Z is $@_i X$, Z is a positive subformula of φ, $\mathcal{M}, \Gamma \nVdash_{h\mathsf{JT}} \langle\!\langle @_i X \rangle\!\rangle_\varphi$, and the result is known for (the positively occurring) X.

Let $X' \in \langle\!\langle X \rangle\!\rangle_\varphi$, so that $@_i X' \in \langle\!\langle @X \rangle\!\rangle_\varphi$. Then $\mathcal{M}, \Gamma \nVdash_{h\mathsf{JT}} @_i X'$, so $\mathcal{M}, \mathcal{V}(i) \nVdash_{h\mathsf{JT}} X'$. Since X' was arbitrary, $\mathcal{M}, \mathcal{V}(i) \nVdash_{h\mathsf{JT}} \langle\!\langle X \rangle\!\rangle_\varphi$, so by the induction hypothesis, $\mathcal{M}, \mathcal{V}(i) \nVdash_{h\mathsf{T}} X$, and hence $\mathcal{M}, \Gamma \nVdash_{h\mathsf{T}} @_i X$.

Negative @ This case is similar to the Positive @ case.

■

The following corollary gives a kind of realization embedding, but not with the simple form we really want. That will come in the next section.

Theorem 5.5 (Weak Realization) *Let φ be a modal formula, indexed. Suppose that φ is* hybrid-*T provable. Then there are members $\varphi_1, \ldots, \varphi_n$ of $\langle\!\langle \varphi \rangle\!\rangle_\varphi$ such that $\varphi_1 \vee \ldots \vee \varphi_n$ is a theorem of* hybrid-*JT (using any axiomatically adequate constant specification).*

Proof Assume $\varphi_1 \vee \ldots \vee \varphi_n$ is not a theorem of *hybrid*-JT for every $\varphi_1, \ldots, \varphi_n \in \langle\!\langle \varphi \rangle\!\rangle_\varphi$. Then $S = \{\neg\varphi' \mid \varphi' \in \langle\!\langle \varphi \rangle\!\rangle_\varphi\}$ is consistent, for otherwise there would be some $\varphi_1, \ldots, \varphi_n \in \langle\!\langle \varphi \rangle\!\rangle_\varphi$ such that $(\neg\varphi_1 \wedge \ldots \wedge \neg\varphi_n) \supset \bot$ would be provable in *hybrid*-JT, and then $\varphi_1 \vee \ldots \vee \varphi_n$ would also be provable, contrary to assumption. Since $S = \{\neg\varphi' \mid \varphi' \in \langle\!\langle \varphi \rangle\!\rangle_\varphi\}$ is consistent, we can construct an S-canonical model in which S is satisfiable. Then there is some world Γ of this canonical model $\mathcal{M}$ at which all members of S are true, and so $\mathcal{M}, \Gamma \not\Vdash_{h\mathsf{JT}} \langle\!\langle \varphi \rangle\!\rangle_\varphi$. Then by Theorem 5.4, $\mathcal{M}, \Gamma \not\Vdash_{h\mathsf{T}} \varphi$, hence φ is not *hybrid*-T provable. ■

5.2 Realization, Finally

We begin by defining something very much like $\langle\!\langle X \rangle\!\rangle_\varphi$, Definition 5.3. This differs from the earlier version in one very significant instance, case 5.

Definition 5.6 Let φ be a formula of *hybrid*-T, indexed. We associate a set, $[\![X]\!]_\varphi$, of formulas with each subformula X of φ.

1. A is an atomic subformula of φ. $[\![A]\!]_\varphi = \{A\}$
2. $X \supset Y$ is a subformula of φ. $[\![X \supset Y]\!]_\varphi = \{A \supset B \mid A \in [\![X]\!]_\varphi$ and $B \in [\![Y]\!]_\varphi\}$
3. $@_i X$ is a subformula of φ. $[\![@_i X]\!]_\varphi = \{@_i A \mid A \in [\![X]\!]_\varphi\}$
4. $\Box X$ is a negative subformula of φ, with index n so that $\Box X = \Box_n X$.
   $[\![\Box X]\!]_\varphi = \{x_n{:}A \mid A \in [\![X]\!]_\varphi\}$.
5. $\Box X$ is a positive subformula of φ.
   $[\![\Box X]\!]_\varphi = \{t{:}A \mid A \in [\![X]\!]_\varphi,\ t$ is any justification term$\}$

Throughout this section we make use of substitutions, replacing justification variables with justification terms. A substitution is a mapping $\sigma = \{x_1/t_1, \ldots, x_n/t_n\}$, where each t_i is different from x_i. Substitution σ maps variable x_i to justification term t_i, and maps x to x for a variable x different from $x_1, \ldots, x_n$. The *domain* of σ is $\{x_1, \ldots, x_n\}$. For a formula X of *hybrid*-JT, the result of applying a substitution σ to it will be denoted $X\sigma$.

If X is a theorem of *hybrid*-JT, and σ is a substitution, $X\sigma$ will also be a theorem. This is easy to see, because substitutions turn axioms into axioms, and rule applications into rule applications. However, the role of constants changes as the result of a substitution. Suppose $\mathcal{C}$ is a constant specification, A is an axiom, and $c_1{:}c_2{:}\ldots c_n{:}A$ is added to a proof using Iterated Axiom Necessitation, where this addition meets constant specification $\mathcal{C}$. Since $A\sigma$ is also an axiom, Iterated Axiom Necessitation allows us to add $c_1{:}c_2{:}\ldots c_n{:}A\sigma$ to a proof, but this may no longer meet specification $\mathcal{C}$. However, a new constant specification, which we can call $\mathcal{C}\sigma$, can be computed from the original one—$c_1{:}c_2{:}\ldots c_n{:}A\sigma \in \mathcal{C}\sigma$ just in case $c_1{:}c_2{:}\ldots c_n{:}A \in \mathcal{C}$. But even if $\mathcal{C}$ was axiomatically appropriate, $\mathcal{C}\sigma$ will not be—axioms that contain variables that σ replaces will generally not have constants assigned to them. But if $\mathcal{C}$ is axiomatically appropriate, $\mathcal{C} \cup \mathcal{C}\sigma$ will be. Consequently, if X is provable using an axiomatically appropriate constant specification, the same will be true for $X\sigma$. From now on we suppress such details.

Definition 5.7 Let φ be a *modal* formula, indexed. We say a substitution σ *lives on* subformula Z of φ provided, for every x in the domain of σ, $x = x_k$ for some indexed necessity occurrence $\Box_k$ in Z. We say σ *lives away from* subformula Z of φ provided, for every x in the domain of σ, $x \neq x_k$ for every indexed necessity occurrence $\Box_k$ in Z. We say σ meets the *no new variable* condition provided, for every x in the domain of σ, the justification term $x\sigma$ contains no variables other than x.

Lemma 5.8 *Assume φ is a modal formula, indexed, and X is a subformula of φ.*

1. *Let σ_Y be a substitution that lives away from X. If $W \in [\![X]\!]_\varphi$ then also $W\sigma_Y \in [\![X]\!]_\varphi$.*
2. *Let σ_X be a substitution that lives on X and σ_Y be a substitution that lives away from X, and assume both substitutions meet the no new variable condition. Then $\sigma_X\sigma_Y = \sigma_Y\sigma_X$.*

Proof Part 1: The proof is by induction on the complexity of X. The atomic case is trivial since no variables are present, and the $\supset$ and @ cases are straightforward. This leaves the two modal cases.

Suppose $\Box X$ is a negative subformula of φ, and the result is known for X. If σ_Y lives away from $\Box X$, then if $x{:}W$ is in $[\![\Box X]\!]_\varphi$, the variable x cannot be in the domain of σ_Y, so $(x{:}W)\sigma_Y = x{:}(W\sigma_Y)$, and the result follows immediately, using the induction hypothesis.

Finally suppose $\Box X$ is a positive subformula of φ, σ_Y lives away from $\Box X$, and the result is known for X. Suppose $t{:}W \in [\![\Box X]\!]_\varphi$. Then $(t{:}W)\sigma_Y = (t\sigma_Y){:}(W\sigma_Y)$. By the induction hypothesis, $W\sigma_Y \in [\![X]\!]_\varphi$, and the result follows because part 5 of Definition 5.6 allows for *any* justification term.

Part 2: Assume the hypothesis, and let x be a variable; we show $x\sigma_X\sigma_Y = x\sigma_Y\sigma_X$.

First, suppose $x = x_k$ for some indexed necessity occurrence $\Box_k$ in X. Since σ_X meets the no new variable condition, the only justification variable that can occur in $x\sigma_X$ is x. Since σ_Y lives away from X, $x\sigma_Y = x$, and so $x\sigma_X\sigma_Y = x\sigma_X$. But also, $x\sigma_Y\sigma_X = x\sigma_X$, hence $x\sigma_X\sigma_Y = x\sigma_Y\sigma_X$.

Second, suppose $x \neq x_k$ for every $\Box_k$ in X. Since σ_X lives on X, $x\sigma_X = x$. And since σ_Y meets the no new variable condition, x is the only variable that can occur in $x\sigma_Y$. Then $x\sigma_Y\sigma_X = x\sigma_Y$, and $x\sigma_X\sigma_Y = x\sigma_Y$, so $x\sigma_X\sigma_Y = x\sigma_Y\sigma_X$ in this case too. ■

Here is the key result that will give us a Realization Theorem, when combined with work from the previous section.

Proposition 5.9 *Let φ be a formula of* hybrid-*T, indexed. For every Z that is a subformula of φ, and for each $Z_1, \ldots, Z_n \in \langle\!\langle Z\rangle\!\rangle_\varphi$, there is a substitution σ and a formula Z' such that:*

1. *$Z' \in [\![Z]\!]_\varphi$*
2. *σ lives on Z and meets the no new variable condition.*
3. *If Z is a positive subformula of φ, $(Z_1 \vee \ldots \vee Z_n)\sigma \supset Z'$ is a theorem of* hybrid-*JT.*
4. *If Z is a negative subformula of φ, $Z' \supset (Z_1 \wedge \ldots \wedge Z_n)\sigma$ is a theorem of* hybrid-*JT.*

Proof By induction on the complexity of Z. If Z is atomic the result is trivial, since $\langle\!\langle Z\rangle\!\rangle_\varphi = [\![Z]\!]_\varphi = \{Z\}$, and we can use the empty substitution. We look at the non-atomic cases in detail.

Positive Implication Suppose Z is $(X \supset Y)$, Z is a positive subformula of φ, $Z_1, \ldots, Z_n \in \langle\!\langle Z \rangle\!\rangle_\varphi$, and the result is known for (negatively occurring) X and (positively occurring) Y.

For each i say $Z_i = (X_i \supset Y_i)$, where $X_i \in \langle\!\langle X \rangle\!\rangle_\varphi$ and $Y_i \in \langle\!\langle Y \rangle\!\rangle_\varphi$. By the induction hypothesis there are substitutions σ_X and σ_Y, living on X and Y respectively and meeting the no new variable condition, and there are $X' \in [\![X]\!]_\varphi$ and $Y' \in [\![Y]\!]_\varphi$ such that $X' \supset (X_1 \wedge \ldots \wedge X_n)\sigma_X$ and $(Y_1 \vee \ldots \vee Y_n)\sigma_Y \supset Y'$ are both provable in *hybrid*-JT. We will show $(Z_1 \vee \ldots \vee Z_n)\sigma \supset Z'$ is provable, where $\sigma = \sigma_X\sigma_Y$, and $Z' = (X'\sigma_Y \supset Y'\sigma_X)$, and both σ and Z' meet the required conditions.

Formulas X and Y are disjoint subformulas of φ, hence have no indexes in common, and so σ_X and σ_Y have disjoint domains. In particular, σ_X lives on X and so lives away from Y, while σ_Y lives on Y and so lives away from X. Let σ be $\sigma_X\sigma_Y = \sigma_Y\sigma_X$ (these are equal by Lemma 5.8). It is easy to see that σ lives on $X \supset Y$ and meets the no new variable condition.

Since $X' \supset (X_1 \wedge \ldots \wedge X_n)\sigma_X$ is provable then also $[X' \supset (X_1 \wedge \ldots \wedge X_n)\sigma_X]\sigma_Y = [X'\sigma_Y \supset (X_1 \wedge \ldots \wedge X_n)\sigma_X\sigma_Y]$ is provable (though the constant specification may change), that is, $X'\sigma_Y \supset (X_1 \wedge \ldots \wedge X_n)\sigma$ is provable. Similarly $(Y_1 \vee \ldots \vee Y_n)\sigma \supset Y'\sigma_X$ is provable. Then the following is provable using only classical logic: $[(X_1 \supset Y_1) \vee \ldots \vee (X_n \supset Y_n)]\sigma \supset (X'\sigma_Y \supset Y'\sigma_X)$. By Lemma 5.8, $X'\sigma_Y \in [\![X]\!]_\varphi$ since $X' \in [\![X]\!]_\varphi$ and σ_Y lives away from X. Likewise $Y'\sigma_X \in [\![Y]\!]_\varphi$. Then $Z' = (X'\sigma_Y \supset Y'\sigma_X) \in [\![X \supset Y]\!]_\varphi$ and we are done.

Negative Implication Suppose Z is $(X \supset Y)$, Z is a negative subformula of φ, $Z_1, \ldots, Z_n \in \langle\!\langle Z \rangle\!\rangle_\varphi$, and the result is known for (positively occurring) X and (negatively occurring) Y.

Again for $i = 1, \ldots, n$ say $Z_i = (X_i \supset Y_i)$, where $X_i \in \langle\!\langle X \rangle\!\rangle_\varphi$ and $Y_i \in \langle\!\langle Y \rangle\!\rangle_\varphi$. This time, by the induction hypothesis there are substitutions σ_X, living on X, and σ_Y, living on Y, both meeting the no new variable condition, and there are $X' \in [\![X]\!]_\varphi$ and $Y' \in [\![Y]\!]_\varphi$ such that both $(X_1 \vee \ldots \vee X_n)\sigma_X \supset X'$ and $Y' \supset (Y_1 \wedge \ldots \wedge Y_n)\sigma_Y$ are provable. Then again, if we set $\sigma = \sigma_X\sigma_Y = \sigma_Y\sigma_X$, the following is provable: $(X'\sigma_Y \supset Y'\sigma_X) \supset [(X_1 \supset Y_1) \wedge \ldots \wedge (X_n \supset Y_n)]\sigma$. Also $X'\sigma_Y \in [\![X]\!]_\varphi$ and $Y'\sigma_X \in [\![Y]\!]_\varphi$, so $(X'\sigma_Y \supset Y'\sigma_X) \in [\![X \supset Y]\!]_\varphi$, and this establishes this case.

Positive Necessity Suppose Z is $\Box X$, Z is a positive subformula of φ, $Z_1, \ldots, Z_n \in \langle\!\langle Z \rangle\!\rangle_\varphi$, and the result is known for (positively occurring) X.

In this case $Z_1, \ldots, Z_n$ are of the form $t_1{:}D_1, \ldots, t_n{:}D_n$, where each t_i is some justification term and D_i is a disjunction of members of $\langle\!\langle X \rangle\!\rangle_\varphi$. Let $D = D_1 \vee \ldots \vee D_n$ be the disjunction of the D_i. D itself is a disjunction of members of $\langle\!\langle X \rangle\!\rangle_\varphi$, so by the induction hypothesis there is some substitution σ, living on X and meeting the no new variable condition, and there is some member $X' \in [\![X]\!]_\varphi$ such that $D\sigma \supset X'$ is provable. Consequently for each i, $D_i\sigma \supset X'$ is provable and so, by Internalization, there is a justification term u_i such that $u_i{:}(D_i\sigma \supset X')$ is provable. But then $(t_i{:}D_i)\sigma \supset (u_i \cdot t_i\sigma){:}X'$ is also provable (using the fact that $(t_i{:}D_i)\sigma = (t_i\sigma){:}(D_i\sigma)$. Let s be the justification term $(u_1 \cdot t_1\sigma) + \ldots + (u_n \cdot t_n\sigma)$. For each i we have the provability of $(t_i{:}D_i)\sigma \supset s{:}X'$, and hence that of $(t_1{:}D_1 \vee \ldots \vee t_n{:}D_n)\sigma \supset s{:}X'$. Since $s{:}X' \in [\![\Box X]\!]_\varphi$, this concludes the positive necessity case.

Negative Necessity Suppose Z is $\Box X$, Z is a negative subformula of φ, $Z_1, \ldots, Z_n \in \langle\!\langle Z \rangle\!\rangle_\varphi$, and the result is known for (negatively occurring) X. Say the occurrence of $\Box$ in $\Box X$ has index k, that is $Z = \Box_k X$.

In this case $Z_1, \ldots, Z_n$ are of the form $x_k : X_1, \ldots, x_k : X_n$, where each $X_i \in \langle\!\langle X \rangle\!\rangle_\varphi$. By the induction hypothesis there is some substitution σ and some $X' \in [\![X]\!]_\varphi$ such that

$X' \supset (X_1 \wedge \ldots \wedge X_n)\sigma$ is provable, where σ lives on X and meets the no new variable condition. We will show $Z' \supset (Z_1 \wedge \ldots \wedge Z_n)\sigma'$ is provable, where $\sigma' = \sigma\{x_k/(s \cdot x_k)\}$, $Z' = x_k{:}X'\{x_k/(s \cdot x_k)\}$, for a particular s, and σ' and Z' meet the required conditions.

For each $i = 1, \ldots, n$, the formula $X' \supset X_i\sigma$ is provable, so by Internalization there is a closed justification term t_i such that $t_i{:}(X' \supset X_i\sigma)$ is provable. Let s be the justification term $t_1 + \ldots + t_n$; then $s{:}(X' \supset X_i\sigma)$ is provable, for each i.

Now consider the substitution $\sigma_0 = \{x_k/(s \cdot x_k)\}$. For each $i = 1, \ldots, n$, $s{:}(X' \supset X_i\sigma)$ is provable, hence so is $[s{:}(X' \supset X_i\sigma)]\sigma_0$. Since s is a closed justification term, $s{:}(X'\sigma_0 \supset X_i\sigma\sigma_0)$ is provable. Then for each i, $x_k{:}X'\sigma_0 \supset (s \cdot x_k){:}X_i(\sigma\sigma_0)$ is provable. Since $\Box_k X$ is a subformula of φ, index k cannot occur as an index in X. Substitution σ lives on X, hence x_k is not in its domain, nor is it introduced by σ since σ meets the no new variable condition. It follows that $x_k(\sigma\sigma_0) = x_k\sigma_0 = (s \cdot x_k)$, and so $[x_k{:}X_i](\sigma\sigma_0) = (s \cdot x_k){:}X_i(\sigma\sigma_0)$. Then for each i, $x_k{:}X'\sigma_0 \supset [x_k{:}X_i](\sigma\sigma_0)$ is provable, and so $x_k{:}X'\sigma_0 \supset [x_k{:}X_i \wedge \ldots \wedge x_k{:}X_n](\sigma\sigma_0)$ is provable.

The substitution σ_0 lives away from X so, since $X' \in [\![X]\!]_\varphi$ then also $X'\sigma_0 \in [\![X]\!]_\varphi$ by Lemma 5.8. If we let Z' be $x_k{:}X'\sigma_0$, we have that $Z' \in [\![\Box_k X]\!]_\varphi$, that is, $Z' \in [\![Z]\!]_\varphi$. And if we let $\sigma' = \sigma\sigma_0$, it is easy to check that σ lives on Z and meets the no new variable condition. And finally, we have verified that $Z' \supset (Z_1 \wedge \ldots \wedge Z_n)\sigma'$ is provable.

Positive @ Suppose Z is $@_i X$, Z is a positive subformula of φ, $Z_1, \ldots, Z_n \in \langle\!\langle Z \rangle\!\rangle_\varphi$, and the result is known for (positively occurring) X.

In this case $Z_1, \ldots, Z_n$ are of the form $@_i X_1, \ldots, @_i X_n$, for some $X_1, \ldots, X_n \in \langle\!\langle X \rangle\!\rangle_\varphi$. By the induction hypothesis there is a substitution σ and a formula $X' \in [\![X]\!]_\varphi$ such that $(X_1 \vee \ldots \vee X_n)\sigma \supset X'$ is provable, and σ lives on X and meets the no new variable condition. Then using @-Necessitation and $K_@$ we have provability of $@_i(X_1 \vee \ldots \vee X_n)\sigma \supset @_i X'$. Since $@_i$ distributes across a disjunction, we have provability of $(@_i X_1 \vee \ldots \vee @_i X_n)\sigma \supset @_i X'$, and this establishes the result in this case.

Negative @ Similar to the preceeding case.

This concludes the proof. ■

A remark on the proof above. It is based on a proof in [8], Proposition 7.8 in the numbering of that paper. The original proof contains an error, manifesting itself in the Positive and Negative Implication parts, and in the Negative Necessity part. That error has been corrected here, and by ignoring nominals the present proof provides a correct proof of the earlier result.

Now realization in a hybrid version follows easily.

Theorem 5.10 (Realization) *If φ is a theorem of* hybrid-*T*, *there is some $\varphi' \in [\![\varphi]\!]_\varphi$ that is a theorem of* hybrid-*JT* *using some axiomatically appropriate constant specification.*

Proof Suppose φ is a theorem of *hybrid*-T. By Theorem 5.5, there are $\varphi_1, \ldots, \varphi_n \in \langle\!\langle \varphi \rangle\!\rangle_\varphi$ such that $\varphi_1 \vee \ldots \vee \varphi_n$ is a theorem of *hybrid*-JT. Then by Proposition 5.9 there is a substitution σ and a formula $\varphi' \in [\![\varphi]\!]_\varphi$ such that $(\varphi_1 \vee \ldots \vee \varphi_n)\sigma \supset \varphi'$ is a theorem of *hybrid*-JT. Since $(\varphi_1 \vee \ldots \vee \varphi_n)\sigma$ is also provable, so is φ'. ■

Note that if $\varphi' \in [\![\varphi]\!]_\varphi$, φ' results from some replacement of $\Box$ occurrences in φ so that negative occurrences are replaced by distinct justification variables. Then the result above really amounts to the usual formulation of Realization.

6 Conclusion

A hybrid/justification logic analog of the modal logic T has been examined. Other modal logics can be treated similarly, though how far this can be pushed remains to be seen. To weaken things to K will require some adjustment, since the analog of $\Box X \supset X$ plays an important role in the proof of Proposition 4.2, but this does not seem to be a major issue. A move to analogs of K4 and S4 seems equally plausible, and probably S5 will not be difficult.

Unfortunately, the hybrid machinery we have been using is that of *basic* hybrid logic—it is not assumed that all possible worlds are named by nominals. It is the move to *named models* that gives hybrid logics much of their interest and power. The machinery for working with named models, proof theoretically, involves the addition of rules to basic hybrid logic. Analogs of these rules can be added to our hybrid/justification logics. The difficulty is in coming up with appropriate and intuitively plausible machinery to allow us to prove an Internalization Theorem, as we did in Theorem 4.3. Almost certainly, once Internalization is achieved other things will fall into place quickly. We hope others will take up this investigation.

Even though we have presented a logic that explicitly combines aspects of both its model theory and its proof theory there is, as yet, no interesting interaction between these. Expressive machinery is present, but nothing much is done with it. One can think of justification logics as being logics of knowledge in which explicit reasons for knowledge can be discussed. One might think that combining such machinery with the ability to 'talk about' states should allow something significant to be said. The language of *hybrid*-JT needs to be explored. What does it allow us to say? This, too, is something we encourage others to think about.

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