INTUITIONISTIC MODEL THEORY AND THE COHEN INDEPENDENCE PROOFS

MELVIN FITTING

Gödel proved the continuum hypothesis consistent with the other axioms of set theory [2] by constructing a transfinite sequence of (domains of) classical logic models $\{M_\alpha\}$, taking a limit L, over all ordinals, and showing it was a model for set theory and the continuum hypothesis (among other things). We will indicate how this procedure can be generalized to transfinite sequences of Saul Kripke's intuitionistic logic models [7] in such a way as to establish the independence results of Cohen [1].

This sort of thing has been done by Vopěnka and others (see refs. [3]–[5] and [10]–[14]) using topological intuitionistic models. Kripke's model structure is closer in form to Cohen's forcing technique, and the methods used are more 'logical'. Neither Vopěnka's nor this method requires countable models for set theory.

First I will briefly sketch Kripke's notion of an intuitionistic logic model, since the notation I use is different from his.

Notation. If P is a function ranging over sets of parameters, by $\hat{P}(\Gamma)$ we mean the collection of all first order formulas with constants from $P(\Gamma)$. $\mathfrak{A}$ is any atomic formula, $\mathfrak{X}$ and $\mathfrak{Y}$ are any formulas.

Definition 1. By an *intuitionistic model* we mean an ordered quadruple $\langle G, \mathrm{R}, \models, P\rangle$, where G is a non-empty set, R is a transitive, reflexive relation on G, $\models$ is a relation between elements of G and formulas, and P is a map from G to non-empty sets of parameters, satisfying for any $\Gamma, \Delta \in G$

(1). $\Gamma \mathrm{R} \Delta \Rightarrow P(\Gamma) \subseteq P(\Delta)$;

(2). $\Gamma \models \mathfrak{A} \Rightarrow \mathfrak{A} \in \hat{P}(\Gamma)$;

(3). $\Gamma \models \mathfrak{A}, \Gamma \mathrm{R} \Delta \Rightarrow \Delta \models \mathfrak{A}$;

(4). $\Gamma \models (\mathfrak{X} \wedge \mathfrak{Y}) \Leftrightarrow \Gamma \models \mathfrak{X}$ and $\Gamma \models \mathfrak{Y}$;

(5). $\Gamma \models (\mathfrak{X} \vee \mathfrak{Y}) \Leftrightarrow (\mathfrak{X} \vee \mathfrak{Y}) \in \hat{P}(\Gamma)$ and $\Gamma \models \mathfrak{X}$ or $\Gamma \models \mathfrak{Y}$,

(6). $\Gamma \models \sim \mathfrak{X} \Leftrightarrow \sim \mathfrak{X} \in \hat{P}(\Gamma)$ and for every $\Delta \in G$ such that $\Gamma \mathrm{R} \Delta$, $\Delta \not\models \mathfrak{X}$;

(7). $\Gamma \models (\mathfrak{X} \supset \mathfrak{Y}) \Leftrightarrow (\mathfrak{X} \supset \mathfrak{Y}) \in \hat{P}(\Gamma)$ and for every $\Delta \in G$ such that $\Gamma \mathrm{R} \Delta$, if $\Delta \models \mathfrak{X}$, $\Delta \models \mathfrak{Y}$;

(8). $\Gamma \models (\exists x)\mathfrak{X}(x) \Leftrightarrow$ for some $a \in P(\Gamma)$, $\Gamma \models \mathfrak{X}(a)$;

(9). $\Gamma \models (\forall x)\mathfrak{X}(x) \Leftrightarrow$ for every $\Delta \in G$ such that $\Gamma \mathrm{R} \Delta$, and for every $a \in P(\Delta)$, $\Delta \models \mathfrak{X}(a)$.

DEFINITION 2. $\mathfrak{X}$ is *valid in the model* $\langle G, \mathrm{R}, \models, P\rangle$ if for every $\Gamma \in G$ such that $\mathfrak{X} \in \hat{P}(\Gamma)$, $\Gamma \models \mathfrak{X}$.

$\mathfrak{X}$ is *valid* if $\mathfrak{X}$ is valid in every model.

THEOREM 1. (Kripke [7]). *$\mathfrak{X}$ is a theorem of intuitionistic logic if and only if $\mathfrak{X}$ is valid.*

The above modeling may be briefly motivated as follows:

G is a collection of possible states of knowledge; any $\Gamma \in G$ may be considered to be a collection of physical facts. $\Gamma \mathrm{R} \Delta$ means if now we know Γ, later we might know Δ. $P(\Gamma)$ is the set of constants constructed by the stage Γ, or the set of parameters introduced in reaching Γ. Finally $\Gamma \models \mathfrak{X}$ means that from the facts Γ we may deduce $\mathfrak{X}$.

If a model $\langle G, \mathrm{R}, \models, P\rangle$ has a countable domain, i.e. $\bigcup\{P(\Gamma) | \Gamma \in G\}$, we may apply Cohen's complete sequence method [1]. Call $H \subseteq G$ an R-*chain* if any two elements are R-comparable. Call H a *complete* R-chain if for any formula $\mathfrak{X}$, only using parameters 'available' in H, for some $\Gamma \in H$, $\Gamma \models \mathfrak{X} \vee \sim \mathfrak{X}$. Then, as in [1], any $\Gamma \in G$ can be included in a complete R-chain. If H is a complete R-chain, $\{\mathfrak{X} |$ for some $\Gamma \in H$, $\Gamma \models \mathfrak{X}\}$ is, if we ignore the universal quantifier, a classical truth set. Now suppose $\mathfrak{X}$ has no universal quantifiers and $\sim\sim\mathfrak{X}$ is not an intuitionistic theorem. The analog of the Skolem-Löwenheim theorem holds for Kripke's models, so for some model with a countable domain $\langle G, \mathrm{R}, \models, P\rangle$ for some $\Gamma \in G$, $\Gamma \not\models \sim\sim\mathfrak{X}$. For some $\Delta \in G$, $\Gamma \mathrm{R} \Delta$ and $\Delta \models \sim\mathfrak{X}$. By the above remarks $\sim\mathfrak{X}$ must belong to some classical truth set, so $\mathfrak{X}$ is not classically valid. (This can be extended to a full proof of Kleene [6] theorem 59).

Suppose we could find some intuitionistic model $\langle G, \mathrm{R}, \models, P\rangle$ in which for some $\Gamma \in G$, $\Gamma \models$ ZF and $\Gamma \not\models \sim\sim$AC, where ZF is the set of Zermelo-Fraenkel axioms, and AC is the axiom of choice, all expressed in (classically equivalent) forms not using the universal quantifier. Then by the above we would have the *classical* independence of the axiom of choice. (Note that $\vdash_{\mathrm{I}}(\mathfrak{X} \supset \sim\sim\mathfrak{Y}) \equiv \sim\sim(\mathfrak{X} \supset \mathfrak{Y})$.) Before showing how this may be done, we present the Gödel construction in order to bring out the analogy.

Let V be a classical Zermelo-Fraenkel model. In [2] Gödel defined over V the sequence $\{M_\alpha\}$ of sets as follows:

$M_0 = \emptyset$,

$M_{\alpha+1}$ is the collection of all definable subsets of M_α,

$M_\lambda = \bigcup_{\alpha<\lambda} M_\alpha$ for limit ordinals λ.

Let L be the class $\bigcup_{\alpha \in V} M_\alpha$. Gödel showed that L was a classical ZF model.

As an introduction to the intuitionistic generalization, we restate the Gödel construction using characteristic functions instead of sets. Now of course '$\in$' is to be considered as a formal symbol, not as set membership.

Let M be some collection and let $\mathfrak{v}$ be a truth function on the set of formulas with constants from M. We say a (characteristic) function f is *definable over* $\langle M, \mathfrak{v}\rangle$ if domain$(f) = M$, range$(f) \subseteq \{\mathrm{T}, \mathrm{F}\}$, and for some formula $\mathfrak{X}(x)$ with one free variable and all constants from M, for all $a \in M$

$$f(a) = \mathfrak{v}(\mathfrak{X}(a)).$$

Let M' be the elements of M together with all functions definable over $\langle M, \mathfrak{v}\rangle$.

We define a truth function $\mathfrak{v}'$ on the set of formulas with constants from M' by defining it for atomic formulas. If $f, g \in M'$ we have three cases:

(1). $f, g \in M$. Let $\mathfrak{v}'(f \in g) = \mathfrak{v}(f \in g)$;

(2). $f \in M$, $g \in M' \dot{-} M$. Let $\mathfrak{v}'(f \in g) = g(f)$;

(3). $f \in M' - M$. Let $\mathfrak{X}(x)$ be the formula which defines f over $\langle M, \mathfrak{v}\rangle$. If there is an $h \in M$ such that $\mathfrak{v}((\forall x)(x \in h \equiv \mathfrak{X}(x))) = \mathrm{T}$ and $\mathfrak{v}'(h \in g) = \mathrm{T}$ let $\mathfrak{v}'(f \in g) = \mathrm{T}$, otherwise let $\mathfrak{v}'(f \in g) = \mathrm{F}$. (Case 3 reduces the situation to case 1 or 2.)

We call the pair $\langle M', \mathfrak{v}'\rangle$ the *derived model of* $\langle M, \mathfrak{v}\rangle$.

Now, let $M_0 = \emptyset$ and let $\mathfrak{v}_0$ be the obvious truth function. Thus we have $\langle M_0, \mathfrak{v}_0\rangle$.

Let $\langle M_{\alpha+1}, \mathfrak{v}_{\alpha+1}\rangle$ be the derived model of $\langle M_\alpha, \mathfrak{v}_\alpha\rangle$.

If λ is a limit ordinal, let $M_\lambda = \bigcup_{\alpha<\lambda} M_\alpha$. Let $\mathfrak{v}_\lambda(f \in g) = \mathrm{T}$ if for some $\alpha < \lambda$, $\mathfrak{v}_\alpha(f \in g) = \mathrm{T}$, otherwise let $\mathfrak{v}_\lambda(f \in g) = \mathrm{F}$. Thus we have $\langle M_\lambda, \mathfrak{v}_\lambda\rangle$.

Let $L = \bigcup_{\alpha \in V} M_\alpha$ and let $\mathfrak{v}(f \in g) = \mathrm{T}$ if for some $\alpha \in V$ $\mathfrak{v}_\alpha(f \in g) = \mathrm{T}$, otherwise let $\mathfrak{v}(f \in g) = \mathrm{F}$. Thus we have the 'class' model $\langle L, \mathfrak{v}\rangle$. All the axioms of ZF will be valid in this model.

Before proceeding to the intuitionistic generalization, we note that it can be shown that for formulas without universal quantifiers it suffices to consider only intuitionistic models with the P-map constant. From now on we will assume this, and we will write the range of the map instead of the map itself. Thus our models now are quadruples $\langle G, \mathrm{R}, \models, S\rangle$, where S is a collection of parameters, etc. Also, for convenience, let B be the collection of all R-closed subsets of G.

Let $\langle G, \mathrm{R}, \models, S\rangle$ be some model. We say a function f is definable over $\langle G, \mathrm{R}, \models, S\rangle$ if domain$(f) = S$, range$(f) \subseteq B$, and for some formula $\mathfrak{X}(x)$ with one free variable, all constants from S and no universal quantifiers, for any $a \in S$

$$f(a) = \{\Gamma | \Gamma \models \mathfrak{X}(a)\}.$$

Let S' be the elements of S together with all functions definable over $\langle G, \mathrm{R}, \models, S\rangle$.

We define a $\models'$ relation by giving it for atomic formulas over S'. If $f, g \in S'$ we have three cases:

(1). $f, g \in S$. Let $\Gamma \models'(f \in g)$ if $\Gamma \models (f \in g)$.

(2). $f \in S$, $g \in S' - S$. Let $\Gamma \models'(f \in g)$ if $\Gamma \in g(f)$.

(3). $f \in S' - S$. Let $\mathfrak{X}(x)$ be the formula which defines f over $\langle G, \mathrm{R}, \models, S\rangle$. Let $\Gamma \models'(f \in g)$ if there is an $h \in S$ such that $\Gamma \models \sim(\exists x)\sim(x \in h \equiv \mathfrak{X}(x))$ and $\Gamma \models'(h \in g)$. (This reduces the situation to case 1 or 2).

We call the model $\langle G, \mathrm{R}, \models', S'\rangle$ the *derived model of* $\langle G, R, \models, S\rangle$.

Now, as above, let V be a classical model for ZF. We define a sequence of intuitionistic models as follows:

Let $\langle G, \mathrm{R}, \models_0, S_0\rangle$ be any intuitionistic model satisfying the following five conditions:

(1). $\langle G, \mathrm{R}, \models_0, S_0\rangle \in V$;

(2). S_0 is a collection of functions such that if $f \in S_0$, domain$(f) \subseteq S_0$ and range $(f) \subseteq B$;

(3). for $f, g \in S_0$, $\Gamma \models_0 (f \in g)$ if and only if $\Gamma \in g(f)$;

(4). (extensionality) for $f, g, h \in S_0$, if $\Gamma \models_0 \sim(\exists x)\sim(x \in f \equiv x \in g)$ and $\Gamma \models_0 \sim\sim(f \in h)$ then $\Gamma \models_0 \sim\sim(g \in h)$;

(5). (regularity) S_0 is well-founded with respect to the relation $x \in$ domain(y).

Remark 1. If we consider the symbols $\vee, \wedge, \sim, \supset, \forall, \exists, (,), \in, x_1, x_2, x_3, \ldots$ to be suitably coded as sets, formulas are sequences of sets, and hence sets. It is in this sense that (1) is meant.

Next, let $\langle G, \mathrm{R}, \models_{\alpha+1}, S_{\alpha+1}\rangle$ be the derived model of $\langle G, \mathrm{R}, \models_\alpha, S_\alpha\rangle$. If λ is a limit ordinal, let $S_\lambda = \bigcup_{\alpha<\lambda} S_\alpha$. Let $\Gamma \models_\lambda (f \in g)$ if for some $\alpha < \lambda$, $\Gamma \models_\alpha (f \in g)$. Thus we have $\langle G, \mathrm{R}, \models_\lambda, S_\lambda\rangle$.

Finally, let $S = \bigcup_{\alpha \in V} S_\alpha$ and $\Gamma \models (f \in g)$ if for some $\alpha \in V$, $\Gamma \models_\alpha (f \in g)$.

Thus we have the 'class' model $\langle G, \mathrm{R}, \models, S\rangle$.

At this point, let me remark without giving the proofs that the techniques which are used in handling the $\{M_\alpha\}$ sequence have their analogues for the above sequences. As in the classical case we may show:

THEOREM 2. *All the axioms of Zermelo–Fraenkel (stated without universal quantifiers) are valid in any such* $\langle G, \mathrm{R}, \models, S\rangle$.

Remark 2. As a special case, let $S_0 = \emptyset$ and let G have one element (then B has two elements, G and $\emptyset$). If we define functions $\mathfrak{v}_\alpha(X) = \{\Gamma \in G | \Gamma \models_\alpha \mathfrak{X}\}$ and we identify G with T and $\emptyset$ with F, the resulting sequence $\langle S_\alpha, \mathfrak{v}_\alpha\rangle$ is identical with the sequence$\langle M_\alpha, v_\alpha\rangle$ above. Thus, as a special case of the above theorem, L is a (classical) ZF model.

At this point it is possible to produce a particular $\langle G, \mathrm{R}, \models_0, S_0\rangle$ with so much symmetry built in that in the resulting class model $\langle G, \mathrm{R}, \models, S\rangle \sim$ AC is valid. From this, as shown above, the classical independence of the axiom of choice follows. Since this model requires a fair amount of detail, rather than give it here I refer you to **[15]** and go on to show how ordinals may be represented in these models.

By putting one more requirement on $\langle G, \mathrm{R}, \models_0, S_0\rangle$ it becomes possible to find quite satisfactory representatives of all the ordinals of V in the class models. Essentially, as in the classical case, each 'ordinal' will be the 'set' of all smaller 'ordinals'. Let us make this more precise.

We use a formula *ordinal*(x) with no universal quantifiers which classically defines the ordinals.

Let us define ordinal representatives as follows: Suppose for each $\beta < \alpha$ we have already defined representatives in S. We call $f \in S$ a *general representative* of α if

(1) if $g \in S$ represents any ordinal $<\alpha$, $(g \in f)$ is valid in $\langle G, \mathrm{R}, \models, S\rangle$.

(2) if for some $\Gamma \in G$, $\Gamma \models (g \in f)$, then for some R-successor Δ of Γ, some $\beta < \alpha$ and some $h \in S$ representing β, $\Delta \models (g = h)$, that is, $\Delta \models \sim(\exists x) \sim(x \in g \equiv x \in h)$.

General representatives would be quite satisfactory to work with, if they existed, even if they were not unique. However, it is convenient to single out canonical representatives.

If f is a general representative of α, we call f a *canonical representative* of α if

(1) for no $g \in$ domain(f) and for no $\Gamma \in G$ does $\Gamma \models (f = g)$;

(2) if $\Gamma \models \sim\sim(g \in f)$, then $\Gamma \models (g \in f)$ for all $g \in$ domain(f).

For reasons to be given in a moment, canonical representatives, if they existed would be delightful to work with. To ensure their existence, we place two requirements on the 0-th models which say essentially that canonical representatives in S_0 (if any) are unique and that there are sufficiently many canonical representatives in S_0 that any element of S_0 which at some point is an ordinal may later be a canonical one.

Formally, call $\langle G, \mathrm{R}, \models_0, S_0\rangle$ *ordinalized* if

(1) no ordinal has more than one canonical representative in S_0;

(2) if $f \in S_0$ and $\Gamma \models_0$ ordinal(f) for some $\Gamma \in G$, then for some R-successor Δ of Γ and some $h \in S_0$ which is a canonical ordinal representative of some ordinal, $\Delta \models_0 (f = h)$.

Now it is not difficult to show the following results:

If $\langle G, \mathrm{R}, \models_0, S_0\rangle$ is ordinalized,

(1) every ordinal of V is uniquely canonically representable by an element of S (denote the representative of α by $\hat{\alpha}$);

(2) $\alpha = \beta$ iff $(\hat{\alpha} = \hat{\beta})$ is valid in $\langle G, \mathrm{R}, \models, S\rangle$,
$\alpha \in \beta$ iff $(\hat{\alpha} \in \hat{\beta})$ is valid in $\langle G, \mathrm{R}, \models, S\rangle$;

(3) ordinal($\hat{\alpha}$) is valid in $\langle G, \mathrm{R}, \models, S\rangle$;

(4) if for some $\Gamma \in G$, $\Gamma \models$ordinal(f), then for some R-successor Δ of Γ and some ordinal α, $\Delta \models (f = \hat{\alpha})$;

(5) if the canonical representative $\hat{\alpha}$ is in $S_{\beta+1} - S_\beta$, $\hat{\alpha}$ is the function defined over the model $\langle G, \mathrm{R}, \models_\beta, S_\beta\rangle$ by the formula ordinal(x).

Again we do not present proofs, but they can be found in [15]. Let me remark that making 0th models ordinalized is a natural requirement; models which are not are rather contrived things.

There is an analogue to the classical notion of absoluteness: Call a formula $\mathfrak{X}(x_1, \ldots, x_n)$ *dominant* if for any $f_1, \ldots, f_n$ in S_α and any $\Gamma \in G$, $\Gamma \models \mathfrak{X}(f_1, \ldots, f_n)$ iff $\Gamma \models_\alpha \mathfrak{X}(f_1, \ldots, f_n)$. Formulas like $(x \in y)$, $(x = y)$ and *ordinal*(x) are dominant, so whether a 0th model is ordinalized or not can be determined by considering it alone.

Let *cardinal*(x) be a formula with no universal quantifiers, which classically defines the cardinals. Then the following may be shown:

THEOREM 3. *Suppose* $\langle G, \mathrm{R}, \models_0, S_0\rangle$ *is ordinalized and for some* $\Gamma \in G$ *and some ordinal* α, $\Gamma \models$ *cardinal*($\hat{\alpha}$). *Then* α *is a cardinal in the model L of constructable sets.*

It is the opposite of this situation that is needed to show the independence of the continuum hypothesis. A proof in Cohen may be adapted to these models to show the following:

$\Gamma, \Delta \in G$ are called *incompatible* if they have no common R-successor. G is called *countably incompatible* if any subset of G of mutually incompatible elements is at most countable in V.

THEOREM 4. *If* $\langle G, R, \models_0, S_0 \rangle$ *is ordinalized, G is countably incompatible, and* α *is a cardinal of V, then cardinal*$(\hat{\alpha})$ *is valid in* $\langle G, \mathrm{R}, \models, S \rangle$.

Now a specific 0th model can be given which produces a class model in which $\sim$(continuum hypothesis) and AC are valid. The model is essentially the same as the one in Cohen and the methods he uses can be adapted.

We remark that constructible set representatives can also be defined and the proof of the independence of the axiom of constructibility of Cohen can be adapted to these models.

I am afraid the foregoing has been only a collection of definitions and results, without proofs. To give the proofs in detail would take pages. To give them in outline is to say they are the analogues of classical proofs or of proofs in Cohen. But now I am going to give even less details than before, only indicating the types of theorems that exist without stating them precisely.

To the best of my knowledge there are three versions of the independence proofs (not counting those above): Cohen's forcing technique, Vopěnka's topological method and Scott and Solovay's Boolean-valued logic approach [9].

The connection between the above intuitionistic methods and those of Cohen should be clear to anyone familiar with Cohen's work.

Since there is a topological model theory for intuitionistic logic, there is of course a topological version of the above. In fact there is a direct translation between Kripke's models and topological models, without going through the respective completeness theorems. I do not know how close translations of the above mentioned proofs would be to those of Vopěnka.

There are also pseudo-Boolean algebra models for intuitionistic logic [8], and again there is a direct translation between Kripke's models and algebraic ones. There are also some connections between pseudo-Boolean and Boolean algebras which apply in this case. Thus the above may be put into the language of Boolean-valued logics. The result is *not* the Scott and Solovay proof. We generalized the $\{M_\alpha\}$ sequence, they generalized the $\{R_\alpha\}$ sequence (sets with rank). Thus two more methods of showing independence become available, a Boolean-valued $\{M_\alpha\}$ sequence and an intuitionistic (or forcing) $\{R_\alpha\}$ sequence. Some details of both may be found in [15].

REFERENCES

[1] P. COHEN, *Set theory and the continuum hypothesis*, W. A. Benjamin, New York (1966).

[2] K. GÖDEL, Consistency proof for the generalized continuum hypothesis, *Proc. Nat. Acad. Sci. U.S.A.* **25** (1939) 220–224.

[3] P. HAJEK and P. VOPĚNKA, Some permutation submodels of the model ∇, *Bull. Acad. Polonaise Sci.* **14** (1966) 1–7.

[4] T. JECH and A. SOCHOR, On Θ model of the set theory, *Bull. Acad. Polonaise Sci.* **14** (1966) 297–303.

[5] T. JECH and A. SOCHOR, Applications of the Θ model, *Bull. Acad. Polonaise Sci.* **14** (1966) 351–355.

[6] S. C. KLEENE, *Introduction to metamathematics*, Van Nostrand, New York (1952).

[7] S. KRIPKE, Semantical analysis of intuitionistic logic I, in *Formal systems and recursive functions*, North-Holland, Amsterdam (1965) 92–130.

[8] H. RASIOWA and R. SIKORSKI, *The mathematics of metamathematics*, Panstwowe Wydawnictwo Naukowe, Warszawa (1963).

[9] D. SCOTT and R. SOLOVAY, Boolean-valued models for set theory, *Summer institute on axiomatic set theory*, Univ. of Cal., Los Angeles, 1967.

[10] P. VOPĚNKA, The limits of sheaves and applications on constructions of models, *Bull. Acad. Polonaise Sci.* **13** (1965) 189–192.

[11] P. VOPĚNKA, On ∇ model of set theory, *Bull. Acad. Polonaise Sci.* **13** (1965) 267–272.

[12] P. VOPĚNKA, Properties of ∇ model, *Bull Acad. Polonaise Sci.* **13** (1965) 441–444.

[13] P. VOPĚNKA, ∇ models in which the generalized continuum hypothesis does not hold, *Bull. Acad. Polonaise Sci.* **14** (1966) 95–99.

[14] P. VOPĚNKA, The limits of sheaves over extremally disconnected compact Hausdorff spaces, *Bull. Acad. Polonaisc Sci.* **15** (1967) 1–4.

[15] M. FITTING, *Intuitionistic logic model theory and forcing*, thesis, Yeshiva Univ., New York (1968); North-Holland, Amsterdam (1969).