

## INTUITIONISTIC RESOLUTION

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In automated theorem proving some form of resolution has been the method of choice for Classical logic. Extensions to other logics have been somewhat awkward because resolution is essentially Classical. The use of clause form is the most obvious manifestation of this, but there are other Classical ties as well. In this paper we present a resolution-style theorem prover for propositional Intuitionistic logic. There is an extension to the first-order case as well, though it is not presented here.

In Classical logic  $X$  and  $\neg X$  are, in a sense, duals of each other. To deny  $X$  is to assert  $\neg X$ , and conversely. This is not the case in Intuitionistic logic where to assert  $X$  is to claim one has a proof of  $X$ . Then to deny this is to claim one does not have a proof of  $X$ . This is far weaker than claiming one has a proof of its negation,  $\neg X$ . The failure of the law of excluded middle is the issue here. The consequence for us is: we must introduce some syntactic mechanism to reflect the proof/no proof duality of Intuitionism. We use *signed* formulas for this, as in [Fitting 1983].

**Definition:** Let  $T$  and  $F$  be two new symbols. By a *signed formula* is meant one of  $TX$  or  $FX$ , where  $X$  is a formula.

Informally, read  $TX$  as "X is provable" and  $FX$  as "X is not provable." The same signs can be used Classically, of course, thinking of  $TX$  as "X is true" and  $FX$  as "X is not true." Classically the signs are not necessary, as observed above.

Clauses traditionally are disjunctions of literals, where a literal now is a signed atomic formula. We will need to broaden that definition, because conversion to an equivalent clause form is not generally available in the Intuitionistic setting. We allow arbitrary signed formulas to occur in clauses, retaining only the disjunctive interpretation. The following definition assumes the notion of Kripke model for Intuitionistic logic is known.

**Definition:** A signed formula  $TX$  is true at a possible world  $\Gamma$  of a Kripke model provided  $X$  is forced at  $\Gamma$ . Likewise  $FX$  is true at  $\Gamma$  provided  $X$  is not forced at  $\Gamma$ .

A *clause* is a finite list  $[Z_1, \dots, Z_n]$  of signed formulas. A clause is true at a possible world  $\Gamma$  provided one of its members is true at  $\Gamma$ .

A *clause set* is a finite list  $[S_1, \dots, S_k]$  of clauses. A clause set is true at a possible world  $\Gamma$  provided each of its members is true at  $\Gamma$ .

When dealing with clauses (and clause sets) we use the following Prolog style notation:  $[X | Y]$  denotes the clause whose first member is the signed formula  $X$ , and  $Y$  is the list consisting of the rest of the signed formulas of the clause. Without loss of generality, we assume no signed formula occurs twice in a clause, hence in  $[X | Y]$ ,  $X$

does not occur in the list Y. Also we tacitly assume that clauses (and clause sets) are sets and so, if X is a member of some clause set we can assume it is the *first* member, and display the set as [X | Y]. Finally, we also use the notation Y\*Z to denote the result of appending the lists Y and Z (and removing repetitions).

Now, the usual resolution rule extends to the non-literal case in a straightforward way.

**Resolution Rule:** A clause set S containing the clauses [TX | A] and [FX | B] may be extended by adding the clause A\*B.

Call a clause set *satisfiable* if it is true at some possible world of some Kripke Intuitionistic model. It is straightforward to show the Resolution Rule above turns a satisfiable clause set into another satisfiable clause set.

Classically, one begins a resolution proof by putting a formula into clause form. Here the steps of such a reduction are incorporated directly into the proof procedure, in the form of certain reduction rules. These rules fall into two classes which we call regular and special.

**Regular reduction rules:** A clause set S that contains the clause shown above the line may have the clause(s) below the line added:

$$\begin{array}{ccc}
 \frac{[\neg X | A]}{[FX | A]} & \frac{[TX \wedge Y | A]}{[TX | A], [TY | A]} & \frac{[FX \wedge Y | A]}{[FX, FY | A]} \\
 \\
 \frac{[TX \vee Y | A]}{[TX, TY | A]} & \frac{[FX \vee Y | A]}{[FX | A], [FY | A]} & \frac{[TX \supset Y | A]}{[FX, TY | A]}
 \end{array}$$

It is straightforward to check that each of these rules preserves satisfiability of clause sets.

**Special reduction rules:** A clause set S that contains the clause shown above the line may have the clause(s) below the line added but first all clauses in S that contain any F-signed formula must be removed. (This includes the clauses displayed above the line in the rules.)

$$\frac{[F\neg X | A]}{[TX | A]} \qquad \frac{[FX \supset Y | A]}{[TX | A], [FY | A]}$$

Each of these rules preserves satisfiability as well, but as the argument is somewhat more involved, we discuss one case.

Suppose S is a satisfiable clause set containing the clause [F¬X|A], and the special rule above is applied; we show the resulting clause set, S\*, is again satisfiable. Say the clauses in S are all true at world Γ of some Kripke model. In particular, [F¬X|A] is true at Γ.

Suppose first that some member of A is true at Γ. That member is still in [TX|A], so that clause set is true at Γ. And the other members of S\* were all in S, so they are all true at Γ. In this case all members of S\* are true at Γ, so S\* is satisfiable.

Next suppose that no member of  $A$  is true at  $\Gamma$ . Since  $[F\neg X|A]$  is true at  $\Gamma$ ,  $F\neg X$  must be true, that is  $\neg X$  is not forced at  $\Gamma$ . By definition of Kripke models there must be a possible world accessible from  $\Gamma$ , call it  $\Delta$ , at which  $X$  is forced. At  $\Delta$ ,  $TX$  is true, hence  $[TX|A]$  is true at  $\Delta$ . If  $C$  is a clause set in  $S$ ,  $C$  is true at  $\Gamma$ . And if  $C$  contains no  $F$ -signed formulas,  $C$  will be true at  $\Delta$  because, in a Kripke model, if a formula is forced at a possible world, it is also forced at any world accessible from it. Again  $S^*$  is satisfiable, but this time at  $\Delta$ .

A *derivation* from a clause set  $S$  is a sequence of clause sets, beginning with  $S$ , each of which comes from the preceding using one of the rules above (or a rule to be stated below). A *proof* of  $X$  is a derivation from the clause set  $[FX]$  ending with a clause set containing the empty clause.

Since the empty clause is not satisfiable, and each rule preserves satisfiability, the existence of a proof for  $X$  means  $FX$  is not true at any world of any Kripke model, in other words,  $X$  is valid in all Kripke models.

**Example:** The following is a proof of  $(X \supset Y) \supset (\neg Y \supset \neg X)$ . We omit the straightforward justification for the steps.

[ $[F((X \supset Y) \supset (\neg Y \supset \neg X))]$ ]  
 [ $[T(X \supset Y), [F(\neg Y \supset \neg X)]]$ ]  
 [ $[T(X \supset Y), [T\neg Y], [F\neg X]]$ ]  
 [ $[T(X \supset Y), [T\neg Y], [TX]]$ ]  
 [ $[T(X \supset Y), [T\neg Y], [TX], [FX, TY]]$ ]  
 [ $[T(X \supset Y), [T\neg Y], [TX], [FX, TY], [FY]]$ ]  
 [ $[T(X \supset Y), [T\neg Y], [TX], [FX, TY], [FY], [TY]]$ ]  
 [ $[T(X \supset Y), [T\neg Y], [TX], [FX, TY], [FY], [TY], [ ]]$ ]

**Remarks:** In Classical resolution one reduces completely to the literal level before beginning to use the Resolution Rule. (One converts to clause form first.) This is still possible here in the limited sense that any use of the Resolution Rule can be converted into uses that only involve signed atomic formulas.

If one removes the deletion-of- $F$  requirement from the Special Reduction Rule cases, a complete proof procedure for Classical logic results.

The Intuitionistic procedure is inherently more complex than Classically because there is a hidden branching involved. If we are in a situation in which Special Reduction Rules can be applied to more than one clause of a clause set, applying a rule to one clause will cause the removal of the other clause. For a proper implementation, if a proof is not found after making such a choice, backtracking to the choice point must occur, and the other choice must be followed up. Of course such choices are independent, and the derivations could be constructed in parallel.

Unfortunately the rules above, though sound, are not complete. With them we are unable to prove the Intuitionistically valid formula  $(P \vee Q) \supset (\neg\neg P \vee \neg\neg Q)$  for instance. We still need the following.

**Special case rule:** Suppose  $[A | R] | S$  is a clause set. (Here  $[A|R]$  is a clause containing the signed formula  $A$ , with  $R$  as the list of the other signed formulas making up the clause.  $S$  is the list consisting of the remaining clauses of the clause set.) If there is a derivation of the empty clause from the clause set  $[A] | S$  then the clause set  $[R | S]$  follows from  $[A | R] | S$ .

Using this rule as well, the following constitutes a proof of  $(P \vee Q) \supset (\neg\neg P \vee \neg\neg Q)$ .

[  $[F(P \vee Q) \supset (\neg\neg P \vee \neg\neg Q)]$  ]  
 [  $[T(P \vee Q), [F\neg\neg P \vee \neg\neg Q]]$  ]  
 [  $[TP, TQ], [F\neg\neg P \vee \neg\neg Q]$  ]  
 [  $[TP, TQ], [F\neg\neg P], [F\neg\neg Q]$  ] (\*)

Now we consider the *case*:

[  $[TP], [F\neg\neg P], [F\neg\neg Q]$  ]  
 [  $[TP], [T\neg P]$  ]  
 [  $[TP], [T\neg P], [FP]$  ]  
 [  $[TP], [T\neg P], [FP], [ ]$  ]

Having derived the empty clause from this case, by the Special Case Rule proving (\*) reduces to deriving the empty clause from [  $[TQ], [F\neg\neg P], [F\neg\neg Q]$  ] which goes as follows.

[  $[TQ], [F\neg\neg P], [F\neg\neg Q]$  ]  
 [  $[TQ], [T\neg Q]$  ]  
 [  $[TQ], [T\neg Q], [FQ]$  ]  
 [  $[TQ], [T\neg Q], [FQ], [ ]$  ].

The system of rules is now complete for Intuitionistic Propositional Logic. A proof can be constructed using an Intuitionistic version of the Model Existence Theorem [Fitting 1973], [Fitting 1983].

### Bibliography:

[Fitting 1973] Model existence theorems for modal and intuitionistic logics, *Journal of Symbolic Logic*, (38) 613-627.

[Fitting 1983] *Proof Methods for Modal and Intuitionistic Logics*, D. Reidel Publishing Company, Dordrecht.