Kleene's Logic, Generalized

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Abstract

Kleene's well-known strong three-valued logic is shown to be one of a family of logics with similar mathematical properties. These logics are produced by an intuitively natural construction. The resulting logics have direct relationships with bilattices. In addition they possess mathematical features that lend themselves well to semantical constructions based on fixpoint procedures, as in logic programming.

1 Introduction

Kleene's strong three-valued logic is among the best-known and best-motivated of the multiple-valued logics, [14]. In this paper we show that it is one of a natural family of multiple-valued logics, all with similar motivations and properties. We begin with a brief look at the Kleene logic itself, and some previous generalizations of it.

One can think of Kleene's three values as *false*, *true* and $\perp$ (*unknown*). This informal reading suggests two natural orderings, concerning 'amount of knowledge' and 'degree of truth.' If we do not know the (classical) truth value of some sentence, then it is possible that an increase in our knowledge could allow us to conclude the sentence is true, or equally well, that it is false. Thus we have a 'knowledge' ordering in which $\perp$ is below both *false* and *true*. On the other hand, if degree of truth is the issue, not knowing a classical truth value to assign to a sentence is better than knowing the sentence is false, while knowing it is true is better yet. Then in the 'truth' ordering, *false* is less than $\perp$ which is less than *true*. Both these orderings are displayed in the double Hasse diagram of Figure 1.

The truth ordering of Figure 1, $\leq_t$, provides us with a complete lattice, and the meet and join are exactly the $\wedge$ and $\vee$ of Kleene's logic. The knowledge ordering, $\leq_k$, does not give us a complete lattice, but it does yield a complete semi-lattice: arbitrary meets exist, while joins exist for directed

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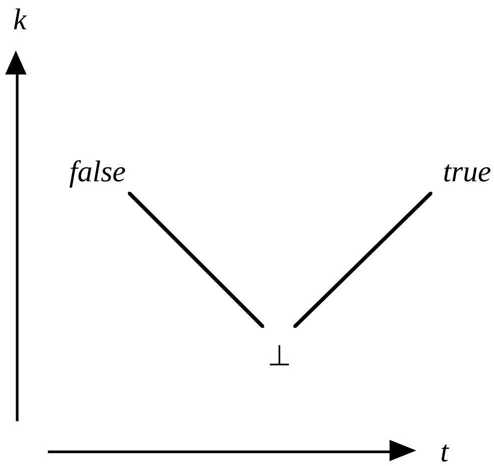

Figure 1: Kleene's Strong 3-Valued Logic

families. Although this is a trivial observation for the Kleene logic, it turns out to be a crucial point for the development of Kripke's theory of truth, see [15] and also [5]. In [4] I extended the technique to provide a semantics for logic programming with negation. The idea, stated simply, was to develop a fixpoint semantics in which one uses the $\leq_t$ ordering to give meaning to the connectives, and the $\leq_k$ ordering to determine the *least* fixed point.

Kleene's logic was, in a sense, completed in [1] by the addition of a fourth truth value, *overdefined*, denoted here by $\top$ (see also [2] and [3] where the four-valued system first appears). The result is one with both the $\leq_t$ and the $\leq_k$ orderings giving the structure of a complete lattice. This logic is displayed in Figure 2. As was shown in [19], the Kripke theory of truth extends naturally from the Kleene logic to the Belnap logic, and it, too, plays a role in logic programming, [7].

The Belnap logic, in turn, found a vast generalization in the family of *bilattices*, due to Matt Ginsberg, [13]. Bilattices are multiple-valued logics with two orderings (and consequently, two sets of connectives), with certain relationships postulated between the orderings. The narrowest class that has been studied is that of *distributive* bilattices — it is also the class with the nicest properties mathematically.

Definition 1.1 a *pre-bilattice* is a structure, $\langle \mathcal{B}, \leq_t, \leq_k \rangle$ where $\mathcal{B}$ is a non-empty set (of truth values) and $\leq_t$ and $\leq_k$ are partial orderings giving $\mathcal{B}$ the structure of a complete lattice. We use $\wedge$ and $\vee$ for meet and join under $\leq_t$, and $\otimes$ and $\oplus$ for meet and join under $\leq_k$.

The operations $\wedge$ and $\vee$ are intended to be the natural generalizations of their classical counterparts. The other two connectives are less familiar, but still of interest. If we think of $\leq_k$ as being an ordering by knowledge, then $\otimes$ is a *consensus* operator: $p \otimes q$ is the most that p and q can agree on. Likewise $\oplus$ is a 'gullability' operator: $p \oplus q$ accepts and combines the knowledge of

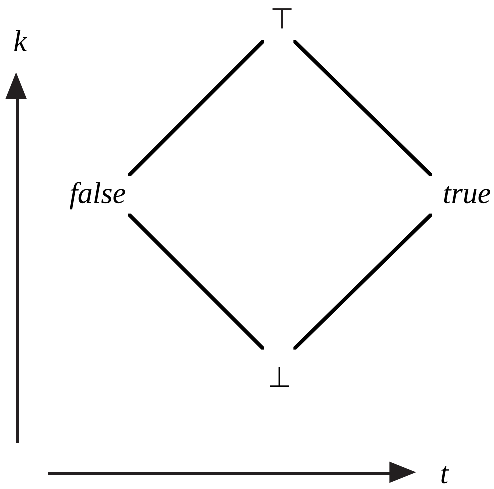

Figure 2: Belnap's 4-valued Logic

p with that of q, whether or not there is a conflict. Loosely, it believes whatever it is told. Since a pre-bilattice has four operations, there are twelve possible distributive laws.

Definition 1.2 A *distributive bilattice* is a pre-bilattice in which all distributive laws hold.

Belnap's four-valued logic is the simplest non-trivial example of a distributive bilattice. There are many others — indeed, there is an intuitively appealing way of constructing them, due to Ginsberg. A description can be found in [13], and also in [9]. Also Kripke's approach to the theory of truth generalizes to allow distributive bilattices as the truth value space, see [6]. Logic programming admits of a similar generalization, see [10] and [8].

The Kleene logic occurs as a natural sublogic of the Belnap one. Indeed, for many bilattices there is a natural way of separating out a sublogic with properties analogous to those of the Kleene logic — we choose those truth values that are, in a certain sense, consistent. Details can be found in [8]; we do not go into it here. But this means we have a way of constructing a whole family of logics related to that of Kleene: use Ginsberg's method to construct distributive bilattices, and for those to which the technique applies, identify the sublogic of 'consistent' truth values. This, indeed, gives us an infinite family of examples (see [9]). But, there is an unsatisfying feature to all this: we are producing Kleene-like logics by producing Belnap-like logics, then throwing part of them away. What we are after now is a more direct construction of Kleene-like logics. Indeed, we will see below that the direct construction even gives us a bigger family of logics to work with — a pleasant side-effect.

2 Simple extensions

Before presenting the family of Kleene-like logics in full generality, we look at a natural subfamily that can serve both as motivation for what will come and as a family of interest in its own right. We begin with a special case of particular interest.

Suppose we are attempting to assign, not classical truth values, but *probability estimates* to formulas. That is, we want to map formulas to values in the unit interval, $[0,1]$. We may be ignorant of some facts, so we may only be able to say the value of a formula lies in the sub-interval $[a,b]$, without being more specific. Indeed, one can make a case that in most circumstances this is the best we will be able to do. Suppose, then, that we take as our truth values such closed intervals: $\mathcal{B} = \{[a,b] \mid 0 \le a \le b \le 1\}$.

Sandewall, in [17], suggested that such truth values be ordered by set inclusion. If $[c,d] \subseteq [a,b]$ then the knowledge inherent in $[c,d]$ is greater than that in $[a,b]$, because $[c,d]$ is narrower. Thus we set $[a,b] \le_k [c,d]$ if $[c,d] \subseteq [a,b]$. Likewise Scott suggested, in [18], that such generalized truth values should be partially ordered by 'degree of truth' so as to yield a lattice. In this case there is a natural way of doing so, set $[a,b] \le_t [c,d]$ if $a \le c$ and $b \le d$. Then, as with bilattices, we have a structure with two associated partial orderings.

It is easy to check that the $\le_t$ ordering gives us a complete lattice. Indeed, in it $[a,b] \wedge [c,d] = [\min\{a,c\}, \min\{b,d\}]$ and $[a,b] \vee [c,d] = [\max\{a,c\}, \max\{b,d\}]$. Under this ordering, $[0,0]$ is least, and $[1,1]$ is greatest; we can identify them with *false* and *true* respectively. On the other hand, the $\le_k$ ordering does not give us a complete lattice — it does, however, yield a complete semilattice. In it, meet is always defined, and $[a,b] \otimes [c,d] = [\min\{a,c\}, \max\{b,d\}]$. But join is not always meaningful; when it is, $[a,b] \oplus [c,d] = [\max\{a,c\}, \min\{b,d\}]$. There is a least member, $[0,1]$, which we denote by $\perp$, but no greatest one. Although this example was explicitly considered in Ginsberg, [13], it does not yield a bilattice in his sense, and so served only as informal motivation. We want to consider it, and generalizations of it, for their own sakes.

As a first attempt at generalizing the construction above, we observe that sometimes we are not interested in the entire unit interval, but only discrete portions of it. For instance, we might want to restrict our attention to just the probabilities $\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$, as being enough for our purposes. In such a case we can still carry out the construction outlined above, and the conclusions still hold. The analog of closed subintervals always yields a truth value space on which two orderings, $\le_k$ and $\le_t$ can be defined as above, with $\le_t$ yielding a complete lattice, and $\le_k$ a complete semi-lattice. But now, specific restrictions on probabilities yield familiar results.

Suppose we make the most extreme restriction: we only consider the two probabilities 0 and 1. In this case there are three intervals: $[0,0] = \{0\}$, $[0,1] = \{0,1\}$, and $[1,1] = \{1\}$, which are identified with *false*, $\perp$, and *true* respectively. In fact, it is easy to see that what we get is the structure of Kleene's three-valued logic, as in Figure 1.

Now consider the obvious next step: use the probabilities 0, $\frac{1}{2}$ and 1. In this case we get six subintervals. There are three that are maximal in the $\le_k$ ordering: $[0,0]$ or *false*, $[\frac{1}{2}, \frac{1}{2}]$, and $[1,1]$ or *true*. There is one that is smallest in the $\le_k$ ordering, $[0,1] = \{0, \frac{1}{2}, 1\}$ or $\perp$. Finally there are two intermediate values, $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. The structure is shown in Figure 3.

In [12] Garcia and Moussavi presented a six-valued logic for representing incomplete knowledge,

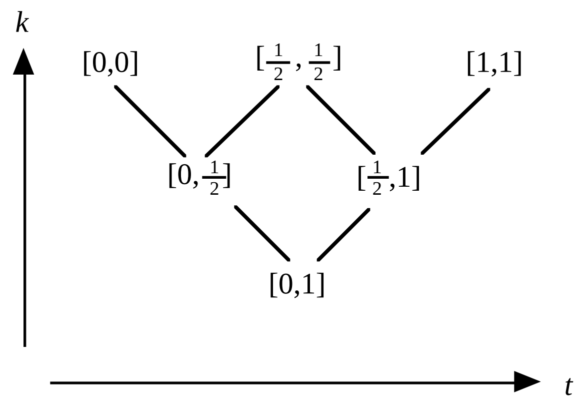

Figure 3: A Six-valued logic

which they thought of as combining both Kleene's and Lukasiewicz's three-valued logics. This work is continued in [11]. The motivation is essentially as follows. Suppose we have two notions of unknown. One, denoted u, represents permanently unknown; a conventional truth value will never be determined. The other, denoted k, represents a "temporary lack of knowledge which is expected to be resolved within the system's time-space bounds." Garcia and Moussavi suggest thinking of the Kleene three-valued logic as 'underneath', with its truth values being $\{false, u, true\}$. Then think of the truth values of the logic under construction as being certain sets of these. There are the singleton sets, $\{false\}$, $\{u\}$ and $\{true\}$, of course. Further, it is suggested we think of k as corresponding to the entire set, $\{false, u, true\}$, so a value of k signifies that we do not know whether a formula is *false*, *true*, or permanently unknown, u. Given this motivation, it quickly becomes apparent that the four truth values thus far constructed are not enough. For instance, $u \wedge k$ should correspond to the set resulting from conjuncting, in Kleene's logic, all members of $\{u\}$ with all members of $k = \{false, u, true\}$, and this turns out to be $\{u, false\}$, which must be taken as a fifth truth value. Similarly $u \vee k$ yields a sixth, $\{u, true\}$. At this point, the system closes.

Garcia and Moussavi go on to consider two orderings of their six truth values, following the motivation of [1]. An ordering by degree of truth is induced by the definitions of $\wedge$ and $\vee$ sketched above; this is a complete lattice. Likewise, an ordering based on knowledge is considered, essentially amounting to reverse subset.

It is straightforward to check that the Garcia-Moussavi logic is exactly that of Figure 3, except for designation of truth values. Their k corresponds to our $[0, 1]$, representing complete lack of information. Likewise their u corresponds to our $[\frac{1}{2}, \frac{1}{2}]$, a truth value which is maximal in the $\leq_k$ ordering, though it is intermediate in the $\leq_t$ ordering. Their knowledge and truth orderings likewise coincide with those of Figure 3.

Instead of beginning with $\{0, \frac{1}{2}, 1\}$, we could have started with any discrete subset of the unit interval that included 0 and 1. For instance, if we begin with $\{0, \frac{1}{3}, \frac{2}{3}, 1\}$ we get a ten-valued logic. In this way we can construct a whole family of generalizations of Kleene's three-valued logic, all with direct motivations. Further, these all share several useful mathematical features. But rather than going into this point here, we postpone it until we have presented a further generalization.

3 The general construction

We have been constructing multiple-valued logics from subsets of $[0,1]$, which is a linear ordering. Now we will extend things to allow more general orderings. This gives rise to two problems: what is the natural analog of the underlying linear ordering, and what is the analog of an interval. The answer to the first question is rather simple. The facts we used above about $[0,1]$, for instance, were things like closure under max and min. Likewise, if we had wanted to treat quantifiers we would have needed the infinitary analogs of these operations, sup and inf. This suggests that what we need of an ordering that is not linear is that it be a *complete lattice*, and this is the assumption we will make from now on. The second question, what is the analog of an interval, requires a little more consideration however.

Let L be a complete lattice, with ordering $\leq_L$. Once we have chosen a family of subsets called intervals, we will give it two orderings. One, $\leq_k$, is quite simple: reverse inclusion. The other, $\leq_t$, is more complex. If I_1 and I_2 are two intervals, whatever we may mean by that, it seems most natural to say I_2 represents greater truth, or less falsehood than I_1 if: for each $x \in I_1$ there is some $y \in I_2$ with $x \leq y$, and for each $y \in I_2$ there is some $x \in I_1$ with $x \leq y$; under these circumstances we set $I_1 \leq_t I_2$. This coincides with the $\leq_t$ ordering used in the previous section, and extends it plausibly to lattice orderings.

Once we have the $\leq_t$ ordering, we have a corresponding equivalence relation: $I_1 \equiv_t I_2$ if $I_1 \leq_t I_2$ and $I_2 \leq_t I_1$. Whatever we take for our notion of interval in L, intervals that are equivalent in this sense should be identical, and this leads us to one part of the definition of interval. Suppose we call a subset S of L *closed* provided it contains, with any two comparable points, all points between them. That is, S is closed if, for each $x, y \in S$ with $x \leq_L y$, if $x \leq_L z \leq_L y$ then $z \in S$. And suppose we define the *closure* of a set S to be the smallest closed superset of S; denote it $cl(S)$. Then it is straightforward to show that, for any subset S of L, $S \equiv_t cl(S)$. Consequently we require, of intervals, that they be closed subsets of L.

Being closed is not enough, however. We want the 'logical' operations on intervals to have simple characterizations. For instance, if I_1 and I_2 are intervals, we want to have a conjunction operation such that $I_1 \wedge I_2 = \{x \wedge y \mid x \in I_1 \text{ and } y \in I_2\}$, where $x \wedge y$ is computed in the underlying lattice L. But we also want $\leq_t$ to give to intervals the structure of a lattice, and we want this conjunction operation to be the meet of that lattice. We now proceed to check the conditions.

First, $I_1 \wedge I_2 \leq_t I_1$ (and similarly for I_2). The argument is as follows. If $x \in I_1 \wedge I_2$ then $x = i_1 \wedge i_2$ for some $i_1 \in I_1$ and $i_2 \in I_2$. Then $x \leq_L i_1$ with $i_1 \in I_1$. On the other hand, if $y \in I_1$, choose any $i_2 \in I_2$. Then $y \wedge i_2 \in I_1 \wedge I_2$, and $y \wedge i_2 \leq_L y$. Thus we have shown that $I_1 \wedge I_2 \leq_t I_1$. (Notice, incidentally, that the argument required that I_2 be non-empty — we chose i_2 from it. This adds a condition to the definition of interval, non-emptyness.)

Second, suppose that $I \leq_t I_1$ and $I \leq_t I_2$. We must show that $I \leq_t I_1 \wedge I_2$. If $x \in I$, $x \leq_L i_1$ for some $i_1 \in I_1$, and $x \leq_L i_2$ for some $i_2 \in I_2$, since $I \leq_t I_1$ and $I \leq_t I_2$. Then $x \leq_L i_1 \wedge i_2$, where $i_1 \wedge i_2 \in I_1 \wedge I_2$. To finish the argument, suppose $y \in I_1 \wedge I_2$. Then $y = i_1 \wedge i_2$ for some $i_1 \in I_1$

and $i_2 \in I_2$. Again, since $I \leq_t I_1$ and $I \leq_t I_2$, there are $a, b \in I$ with $a \leq_L i_1$ and $b \leq_L i_2$. The argument could be completed if we knew that $a \wedge b \in I$, since $a \wedge b \leq_L i_1 \wedge i_2 \leq_L y$. Thus we are led to impose one more condition on intervals: they should be closed under the meet operation, $\wedge$, of L.

Similar considerations lead us to impose closure under $\vee$ as well, and so intervals should, themselves, be sublattices of L. Further, when we come to take infinitary meets and joins into account we are led to requiring closure under them as well, and so the requirement becomes: intervals should be complete lattices themselves.

Summing things up so far, an interval in L is a closed, non-empty subset of L that is a complete lattice. But now that all the pieces are assembled, the definition can be considerably simplified. If I meets these conditions, since it is a complete lattice, it must contain its inf, say a, and its sup, say b. Since it is closed, it must contain all members of L between a and b as well. On the other hand, for any $a, b \in L$ with $a \leq_L b$, the set $\{x \in L \mid a \leq_L x \leq_L b\}$ is easily seen to meet all the conditions we have decided to require of an interval. Consequently, we are finally led to the following much simplified definition.

Definition 3.1 Let $a, b \in L$ with $a \leq_L b$. The *interval* determined by a and b, denoted $[a, b]$, is $\{x \in L \mid a \leq_L x \leq_L b\}$.

Certainly if one were naively to generalize the notion of interval from linear orderings to lattices, this is very likely the definition that would result. But then one might wonder whether it is sufficient — whether some more general notion might be better yet. The point of the work above is that the naive notion is also the natural one in this case.

Note From now on, we will only use the notation $[a, b]$ for intervals, and so there is the tacit assumption that $a \leq_L b$.

Definition 3.2 Let L be a complete lattice. $\mathcal{K}(L)$ is the structure $\langle \mathcal{I}(L), \leq_k, \leq_t \rangle$ where $\mathcal{I}(L)$ is the set of all intervals in L; and for $[a, b], [c, d] \in \mathcal{I}(L)$, $[a, b] \leq_k [c, d]$ if $[c, d] \subseteq [a, b]$; and $[a, b] \leq_t [c, d]$ if for each $x \in [a, b]$ there is some $y \in [c, d]$ with $x \leq_L y$ and for each $y \in [c, d]$ there is some $x \in [a, b]$ with $x \leq_L y$.

We will continue the notation used in the previous section: $\wedge$ and $\vee$ are meet and join under $\leq_t$, and *false* and *true* are smallest and biggest members under this ordering; $\otimes$ and $\oplus$ are meet and join under $\leq_k$, and $\perp$ is least member under it. In general we will use 0 and 1 for smallest and biggest members of L, so that *false* $= [0, 0]$, *true* $= [1, 1]$ and $\perp = [0, 1]$.

4 Examples

We will see, later on, that a critical point concerning the underlying lattice is whether or not it has an order reversing involution. The linear examples above all do. In this section we present two examples involving lattices that are not linear, one with this property, one without.

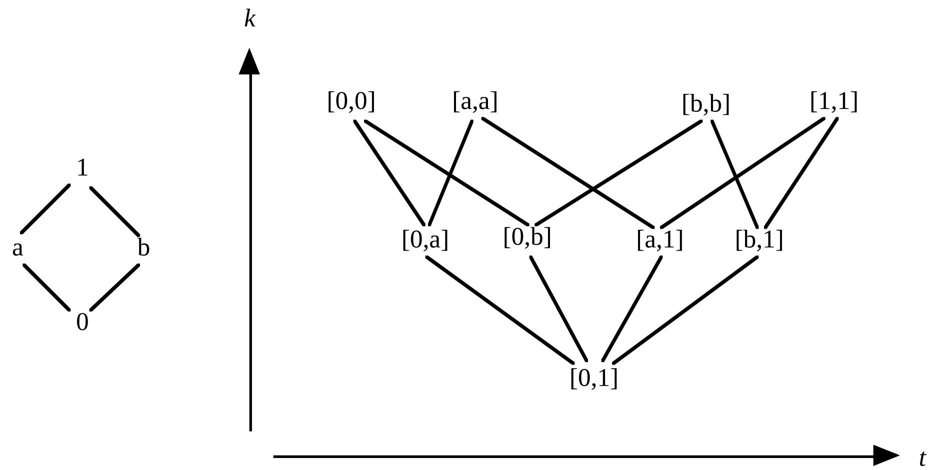

Figure 4: Vertically Symmetric L, and $\mathcal{K}(L)$

The first example is given in Figure 4. The underlying lattice L has four points, while $\mathcal{K}(L)$ has nine. In this example, L does have an order reversing involution, interchange 0 and 1 while leaving a and b alone. We refer to this as 'vertical symmetry.'

For an intuitive motivation of the structures in Figure 4, consider the following. Suppose we have two experts, A and B, and we ask each of them a yes/no question. Interpret the four points of L as follows. The point 0 represents a 'no' answer from both; a represents a 'yes' from A and a 'no' from B; b represents a 'yes' from B and a 'no' from A; and 1 represents a 'yes' from both. Then in $\mathcal{K}(L)$, the interval $[0, a]$, for instance, represents the state of partial knowledge in which the answer of A is not known but B has answered 'no.' An increase in knowledge will either raise us to $[0, 0]$, if A answers 'no,' or to $[a, a]$ if A answers 'yes.' Likewise $[a, 1]$ represents the state of partial knowledge in which A has answered 'yes' but the answer of B is not known.

The next example, shown in Figure 5, is more complicated, and does not involve vertical symmetry. We do not supply an informal application for this one.

5 Basic results

The definitions of $\leq_k$ and particularly of $\leq_t$ are more complicated than need be. The complexity came from our attempts to be as general as possible. Now that we have a simple definition of interval, it is easy to simplify the characterization of the orderings as well. The following Proposition has a straightforward proof, which we omit.

Proposition 5.1 *Let $[a, b]$ and $[c, d]$ be intervals in $\mathcal{K}(L)$. Then:*

1. $[a, b] \leq_t [c, d]$ *if and only if* $a \leq_L c$ *and* $b \leq_L d$*,*

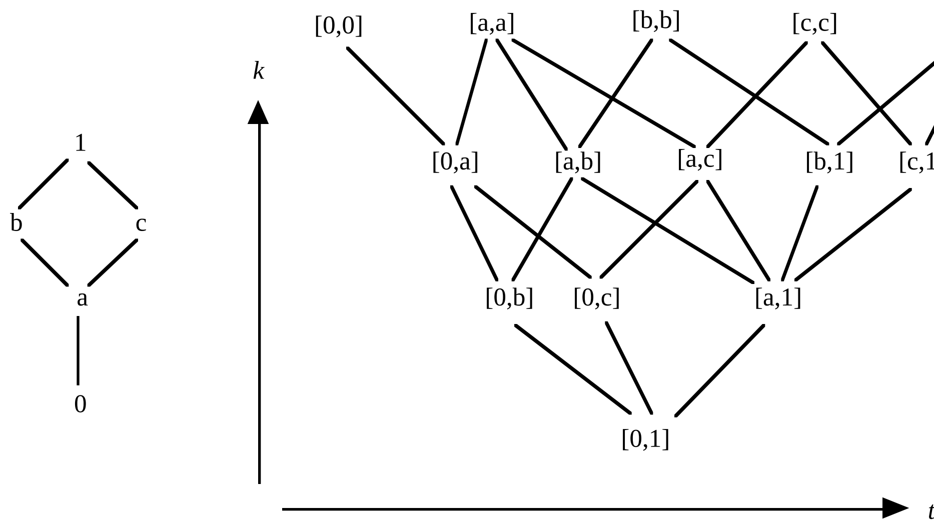

Figure 5: L Without Vertical Symmetry, and $\mathcal{K}(L)$

2. $[a,b] \leq_k [c,d]$ *if and only if* $a \leq_L c$ *and* $d \leq_L b$.

Next, the four basic operations of $\mathcal{K}(L)$, meet and join under each of $\leq_t$ and $\leq_k$, also have simple characterizations in terms of the meet and join of L.

Proposition 5.2 *Again let* $[a,b]$ *and* $[c,d]$ *be intervals in* $\mathcal{K}(L)$. *Then:*

1. $[a,b] \wedge [c,d] = [a \wedge c, b \wedge d]$,
2. $[a,b] \vee [c,d] = [a \vee c, b \vee d]$,
3. $[a,b] \otimes [c,d] = [a \wedge c, b \vee d]$,
4. $[a,b] \oplus [c,d] = [a \vee c, b \wedge d]$ *(when defined).*

Proof We check only item 3, the rest are similar. First, $a \wedge c \leq_L a$ and $b \leq_L b \vee d$, so by the previous Proposition, $[a \wedge c, b \vee d] \leq_k [a,b]$. Similarly $[a \wedge c, b \vee d] \leq_k [c,d]$. It follows that $[a \wedge c, b \vee d] \leq_k [a,b] \otimes [c,d]$.

Next, suppose $[u,v] \leq_k [a,b]$ and $[u,v] \leq_k [c,d]$. Then $u \leq_L a$ and $u \leq_L c$, so $u \leq_L a \wedge c$. Similarly $b \leq_L v$ and $d \leq_L v$ so $b \vee d \leq_L v$. Then $[u,v] \leq_k [a \wedge c, b \vee d]$. It follows that $[a,b] \otimes [c,d] \leq_k [a \wedge c, b \vee d]$. □

The operations $\otimes$ and $\oplus$ are defined using the $\leq_k$ ordering, but they behave well with respect to the $\leq_t$ ordering, as well, and similarly for $\wedge$ and $\vee$ and $\leq_k$.

Proposition 5.3 (The Interlacing Conditions) *For* $[a,b], [c,d]$ *and* $[e,f]$ *intervals in* $\mathcal{K}(L)$*:*

1. *if* $[a,b] \le_k [c,d]$ *then* $[a,b] \wedge [e,f] \le_k [c,d] \wedge [e,f]$,
2. *if* $[a,b] \le_k [c,d]$ *then* $[a,b] \vee [e,f] \le_k [c,d] \vee [e,f]$,
3. *if* $[a,b] \le_t [c,d]$ *then* $[a,b] \otimes [e,f] \le_t [c,d] \otimes [e,f]$,
4. *if* $[a,b] \le_t [c,d]$ *then* $[a,b] \oplus [e,f] \le_t [c,d] \oplus [e,f]$ *(when defined).*

Proof We only show item 2 as a representative. Suppose $[a,b] \le_k [c,d]$, so that $a \le_L c$ and $d \le_L b$. Then $a \wedge e \le_L c \wedge e$ and $d \vee f \le_k b \vee f$. It follows that $[a \wedge e, b \vee f] \le_k [c \wedge e, d \vee f]$ and so $[a,b] \otimes [e,f] \le_t [c,d] \otimes [e,f]$. □

The corresponding interlacing conditions for bilattices have played an important role in both their theory and their applications (see [6], [10], and [16]). Stronger still are distributivity laws. Since there are four operations available, there are twelve distributive laws that are possible. The following proposition has a straightforward proof, which we omit.

Proposition 5.4 (Distributivity) *Suppose* L *is a distributive lattice. Then all distributive laws hold in* $\mathcal{K}(L)$*, provided the operations involved are defined.*

6 Relationships with bilattices

The Belnap logic, Figure 2, contains the Kleene logic within it. In a similar way a large number of bilattices contain within them natural generalizations of Kleene's logic (see [9]). In this section we compare these generalizations arising from bilattices with those constructed above, using intervals.

Suppose we have a pre-bilattice $\langle \mathcal{B}, \le_t, \le_k \rangle$. It has a *negation* if there is a mapping $\neg$ from $\mathcal{B}$ to itself that reverses $\le_t$, leaves $\le_k$ alone, and is an involution. In the Belnap logic, for instance, negation is left-right symmetry. Likewise the pre-bilattice has a *convolution* if there is a mapping $-$ from $\mathcal{B}$ to itself that reverses $\le_k$, leaves $\le_t$ alone, and is an involution. In the Belnap logic convolution arises from the vertical symmetry. If both negation and convolution are present, we will generally require that they commute with each other. There are many such examples.

Suppose we call a member b of a pre-bilattice with a convolution *consistent* if $b \le_k -b$. It can be shown that the consistent members always constitute a natural generalization of Kleene's logic, in the following sense. The collection of consistent members must contain *false*, $\bot$ and *true*, and be closed under $\wedge$ and $\vee$ and their infinitary analogs. In addition, they will be closed under $\neg$ if negation is present and commutes with conflation. Finally, they will be closed under $\otimes$ and its infinitary analog, but will only be closed under $\oplus$ for directed families. These are the conditions on Kleene's logic that made it appropriate for use in Kripke's theory of truth, for instance, and a similar theory can be developed based on any of these generalizations. So, the question that concerns us now is: under what circumstances can a Kleene-like logic, created using the interval construction presented earlier, be identified with the consistent members of a bilattice.

There is a simple way of constructing bilattices, due to Ginsberg. Suppose L_1 and L_2 are complete lattices. By $L_1 \odot L_2$ we mean the structure $\langle L_1 \times L_2, \le_t, \le_k \rangle$ where $\langle a,b \rangle \le_t \langle c,d \rangle$ if $a \le_{L_1} c$ and $d \le_{L_2} b$, and $\langle a,b \rangle \le_k \langle c,d \rangle$ if $a \le_{L_1} c$ and $b \le_{L_2} d$. The intuition behind this construction is rather satisfying. Think of L_1 as the lattice of values used to measure the degree of belief we have in a sentence; probabilites, weighted opinions of experts, etc. Likewise think of L_2 as the lattice of values used to measure our degree of doubt. There is no requirement that we assess belief and doubt in the same way. Then a member of $L_1 \times L_2$ embodies an assessment of both belief and doubt. The ordering $\leq_t$ intuitively says 'degree of truth' increases if belief goes up and doubt goes down. Likewise $\leq_k$ says 'degree of knowledge' goes up if both belief and doubt increase. It is easy to check that $L_1 \odot L_2$ will always be a pre-bilattice that satisfies the interlacing conditions, an *interlaced bilattice*. In addition, if both L_1 and L_2 are distributive lattices, $L_1 \odot L_2$ will be a distributive bilattice.

The construction above can be carried a little further. If $L_1 = L_2$ a natural negation can be introduced into $L_1 \odot L_2$: simply set $\neg\langle a, b\rangle = \langle b, a\rangle$. Further, if $L_1 = L_2$ and there is an order reversing involution on L_1 a conflation operation can be introduced that commutes with the negation: set $-\langle a, b\rangle = \langle -b, -a\rangle$, where $-a$ is the involute of a in L_1. All of this is straightforward to check.

Theorem 6.1 *Suppose L is a complete lattice with an order reversing involution. Then $\mathcal{K}(L)$ is isomorphic to the set of consistent members of an interlaced bilattice with negation and conflation.*

Proof Define a mapping $\theta : \mathcal{K}(L) \to L \odot L$ as follows. $[a, b]\theta = \langle a, -b\rangle$, where $-b$ is the involute of b in L. Now, $\langle a, -b\rangle$ is consistent in $L \odot L$ iff $\langle a, -b\rangle \leq_k -\langle a, -b\rangle = \langle b, -a\rangle$ iff $a \leq_L b$ and $-b \leq_L -a$ iff $a \leq_L b$. It follows that $[a, b]$ is an interval in $\mathcal{K}(L)$ if and only if $[a, b]\theta$ is a consistent member of $L \odot L$. That θ is an isomorphism is now straightforward. □

This Theorem applies, for example, to the system of Figure 4, but not to that of Figure 5. For the class of distributive bilattices, the bilattice construction given above is completely general. That is, if $\mathcal{B}$ is a distributive bilattice with a negation and a conflation that commute, then $\mathcal{B}$ is isomorphic to $L \odot L$ where L is a complete, distributive lattice with an order reversing involution. A proof of this can be found in [9], with parts of it appearing in [8] and [13].

Theorem 6.2 *Suppose $\mathcal{B}$ is a distributive bilattice with a negation and a conflation that commute. Then the substructure of consistent members of $\mathcal{B}$ is isomorphic to $\mathcal{K}(L)$ for a complete, distributive lattice L.*

Proof According to the remarks above, $\mathcal{B}$ is isomorphic to $L \odot L$, where L is a complete lattice with an order reversing involution. Then, as in the proof of the preceeding Theorem, define $\theta : \mathcal{K}(L) \to L \odot L$ by $[a, b]\theta = \langle a, -b\rangle$. It is easy to check that this is the desired isomorphism. □

There is an obvious gap between the two Theorems above, which arises from the lack of a good representation theorem for interlaced, as opposed to distributive bilattices. But at any rate, the relationship between the interval construction and the bilattice construction is fully determined for distributive lattices.

7 Conclusion

The interval based construction of a family of logics generalizing Kleene's has an intuitive appeal to it. In addition, it produces a family of logics with a useful mathematical structure. As we remarked above, all such logics will be closed under meets and joins in the $\leq_t$ ordering, and under meets and directed joins in the $\leq_k$ ordering. This means that Kripke's fixpoint machinery, including the notion of *intrinsic* or *optimal* fixpoint, applies with essentially no changes. This machinery is appropriate in logic programming, too [4]. In addition, there are close relationships with bilattices. Consequently we expect that at least some members of the family of Kleene-like logics will find applications. First, of course, issues of axiomatization, and efficient proof procedures need to be addressed. There is clearly much enjoyable work ahead.

References

[1] Belnap, Jr., N. D. A useful four-valued logic. In *Modern Uses of Multiple-Valued Logic*, J. M. Dunn and G. Epstein, Eds. D. Reidel, 1977.

[2] Dunn, J. M. Natural language versus formal language. presented at the Joint APA-ASL symposium, New York, dec 1969.

[3] Dunn, J. M. Intuitive semantics for first-degree entailments and coupled trees. *Philosophical Studies 29* (1976), 149–168.

[4] Fitting, M. C. A Kripke/Kleene semantics for logic programs. *Journal of Logic Programming* (1985), 295–312.

[5] Fitting, M. C. Notes on the mathematical aspects of Kripke's theory of truth. *Notre Dame Journal of Formal Logic 27* (1986), 75–88.

[6] Fitting, M. C. Bilattices and the theory of truth. *Journal of Philosophical Logic 18* (1989), 225–256.

[7] Fitting, M. C. Negation as refutation. In *Proceedings of the Fourth Annual Symposium on Logic in Computer Science* (1989), R. Parikh, Ed., IEEE, pp. 63–70.

[8] Fitting, M. C. Bilattices in logic programming. In *The Twentieth International Symposium on Multiple-Valued Logic* (1990), G. Epstein, Ed., IEEE, pp. 238–246.

[9] Fitting, M. C. Kleene's three-valued logics and their children. In *Proceedings of the Bulgarian Kleene 90 Conference* (1990). Forthcoming.

[10] Fitting, M. C. Bilattices and the semantics of logic programming. *Journal of Logic Programming* (1991). Forthcoming.

[11] Font, J. M., and Moussavi, M. Note on a six-valued extension of three-valued logic. Mathematics Preprint 86, Universitat de Barcelona, November 1990.

[12] Garcia, O. N., and Moussavi, M. A six-valued logic for representing incomplete knowledge. In *Proc. Twentieth Int. Symp. on Multiple-Valued Logic* (1990), G. Epstein, Ed., IEEE Computer Society Press, pp. 110–114.

[13] Ginsberg, M. L. Multivalued logics: a uniform approach to reasoning in artificial intelligence. *Computational Intelligence 4* (1988), 265–316.

[14] Kleene, S. C. *Introduction to Metamathematics*. D. Van Nostrand, 1950.

[15] Kripke, S. Outline of a theory of truth. *The Journal of Philosophy 72* (1975), 690–716. Reprinted in *New Essays on Truth and the Liar Paradox*, R. L. Martin, ed., Oxford (1983).

[16] Romanowska, A., and Trakul, A. On the structure of some bilattices. In *Universal and Applied Algebra* (1990), K. Hatkowsak and Stowski, Eds., World Scientific.

[17] Sandewall, E. A functional approach to non-monotonic logic. In *Proc. of the Ninth Int. Joint Conf. on Artificial Intelligence* (1985), pp. 100–106.

[18] Scott, D. Some ordered sets in computer science. In *Ordered Sets*, I. Rival, Ed. D. Reidel, 1982, pp. 677–718.

[19] Visser, A. Four valued semantics and the liar. *Journal of Philosophical Logic 13* (1984), 181–212.