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Author(s): Melvin Fitting

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## MODEL EXISTENCE THEOREMS FOR MODAL AND INTUITIONISTIC LOGICS

MELVIN FITTING

**§1. Introduction.** In classical logic a collection of sets of statements (or equivalently, a property of sets of statements) is called a *consistency property* if it meets certain simple closure conditions (a definition is given in §2). The simplest example of a consistency property is the collection of all consistent sets in some formal system for classical logic. The *Model Existence Theorem* then says that any member of a consistency property is satisfiable in a countable domain. From this theorem many basic results of classical logic follow rather simply: completeness theorems, the compactness theorem, the Lowenheim-Skolem theorem, and the Craig interpolation lemma among others. The central position of the theorem in classical logic is obvious. For the infinitary logic  $L_{\omega_1\omega}$  the Model Existence Theorem is even more basic as the compactness theorem is not available; [8] is largely based on it.

In this paper we define appropriate notions of consistency properties for the first-order modal logics  $S4$ ,  $T$  and  $K$  (without the Barcan formula) and for intuitionistic logic. Indeed we define two versions for intuitionistic logic, one deriving from the work of Gentzen, one from Beth; both have their uses. Model Existence Theorems are proved, from which the usual known basic results follow. We remark that Craig interpolation lemmas have been proved model theoretically for these logics by Gabbay ([5], [6]) using ultraproducts. The existence of both ultraproduct and consistency property proofs of the same result is a common phenomena in classical and infinitary logic. We also present extremely simple tableau proof systems for  $S4$ ,  $T$ ,  $K$  and intuitionistic logics, systems whose completeness is an easy consequence of the Model Existence Theorems. Indeed, the existence of a ‘good’ tableau proof system for a logic is equivalent to the existence of a ‘useful’ notion of consistency property for the logic (a vague but valid statement). Finally, various embedding theorems (classical in  $S5$ , intuitionistic in classical, classical in  $S4$ ) are proved using classical consistency properties.

We were not able to extend our methods to the modal logics  $B$  and  $S5$ . The symmetry of the accessibility relation of their Kripke models seems difficult to handle. Possibly an approach using “prefixed” formulas as in [4] will work, though we suspect the resulting Model Existence Theorems will be more difficult to apply.

We use the uniform notation of [10] adapted to our needs. This enables us to take all connectives, quantifiers and modal operators as primitive. We first work with  $S4$ , then give modifications to treat  $T$  and  $K$ . Intuitionistic logic is discussed

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last. We begin, however, with a section on classical consistency properties, both for their own sake and to establish notation.

**§2. Classical consistency properties.** We assume we are dealing with a first-order language  $L_C$  built up from countably many constants, variables and relation symbols. By  $L_C^*$  we mean the language which results when the list of constants of  $L_C$  is enlarged by the addition of countably many new constant symbols. We take all of  $\wedge, \vee, \sim, \supset, \forall,$  and  $\exists$  as primitive. We follow the convention that a *statement* is a formula with no free variables. Since we are assuming such a large set of primitives, a uniform treatment, as in [10] becomes advisable. We may use either unsigned or signed statements at this point. We find unsigned statements slightly simpler here, but signed statements will be essential for intuitionistic logic, and will be presented then.

The collection of statements is divided into six groups: atomic statements, negations of atomic statements, conjunctives ( $\alpha$ ), disjunctives ( $\beta$ ), universals ( $\gamma$ ) and existentials ( $\delta$ ). Associated with each conjunctive or  $\alpha$  statement are two *components*,  $\alpha_1$  and  $\alpha_2$ . Likewise, with each disjunctive or  $\beta$  statement are associated its components,  $\beta_1$  and  $\beta_2$ . The following chart defines these notions.

$\alpha$	$\alpha_1$	$\alpha_2$
$(X \wedge Y)$	$X$	$Y$
$\sim(X \vee Y)$	$\sim X$	$\sim Y$
$\sim(X \supset Y)$	$X$	$\sim Y$
$\sim \sim X$	$X$	$X$

$\beta$	$\beta_1$	$\beta_2$
$(X \vee Y)$	$X$	$Y$
$\sim(X \wedge Y)$	$\sim X$	$\sim Y$
$(X \supset Y)$	$\sim X$	$Y$

Associated with each universal or  $\gamma$  statement are its *instances*  $\gamma(c)$  for each constant symbol  $c$ . Likewise for existential or  $\delta$  statements.

$\gamma$	$\gamma(c)$
$(\forall x)A(x)$	$A(c)$
$\sim(\exists x)A(x)$	$\sim A(c)$

$\delta$	$\delta(c)$
$(\exists x)A(x)$	$A(c)$
$\sim(\forall x)A(x)$	$\sim A(c)$

Now we define the notion of classical consistency property. Let  $\mathcal{C}$  be a collection of nonempty sets of statements of  $L_C^*$ .  $\mathcal{C}$  is a *classical consistency property* if, for each  $S \in \mathcal{C}$ ,

- (1) if  $A$  is atomic, not both  $A \in S$  and  $\sim A \in S$ ,
- (2)  $\alpha \in S \Rightarrow S \cup \{\alpha_1, \alpha_2\} \in \mathcal{C}$ ,
- (3)  $\beta \in S \Rightarrow S \cup \{\beta_1\} \in \mathcal{C}$  or  $S \cup \{\beta_2\} \in \mathcal{C}$ ,
- (4)  $\gamma \in S \Rightarrow S \cup \{\gamma(c)\} \in \mathcal{C}$  for each constant  $c$  of  $L_C^*$ ,
- (5)  $\delta \in S \Rightarrow S \cup \{\delta(c)\} \in \mathcal{C}$  for some constant  $c$  of  $L_C^*$ .

**MODEL EXISTENCE THEOREM FOR CLASSICAL LOGIC.** *Let  $S$  be a set of statements of  $L_C$ . If  $S$  belongs to some classical consistency property,  $S$  is satisfiable.*

It is here that we must observe that basically two definitions of classical consistency property have been proposed in the literature. The above, which we will temporarily call *weak* consistency, and one (see [10]) which substitutes for clause (5) the condition

(5')  $\delta \in S \Rightarrow S \cup \{\delta(c)\} \in \mathcal{C}$  for each constant  $c$  not appearing in  $S$ .

Let us call sets  $\mathcal{C}$  satisfying conditions (1)–(4) and (5') *strong consistency properties*. For modal and intuitionistic logics we will need both notions, so we now discuss their relationship in the classical case, which carries over to the nonclassical.

Let  $\mathcal{C}_s$  be a strong consistency property. If we define  $\mathcal{C}_w$  by  $S \in \mathcal{C}_w$  if  $S \in \mathcal{C}_s$  and there are infinitely many constants of  $L_C^*$  not appearing in  $S$ ; it is easy to see that  $\mathcal{C}_w$  is a weak consistency property (and if  $S$  is a set of statements of  $L_C$ , then  $S \in \mathcal{C}_s$  implies  $S \in \mathcal{C}_w$ ).

Let us call  $\sigma$  a *substitution* if  $\sigma$  is a map from the set of constants of  $L_C^*$  to itself. If  $X$  is a formula let  $\sigma(X)$  be the result of applying  $\sigma$  to each constant in  $X$ . Similarly  $\sigma$  may be extended to sets of statements. Now, suppose  $\mathcal{C}_w$  is a weak consistency property. Define  $\mathcal{C}_s$  by  $S \in \mathcal{C}_s$  if, for some substitution  $\sigma$ ,  $\sigma(S) \in \mathcal{C}_w$ . Then  $\mathcal{C}_s$  is a strong consistency property (and  $\mathcal{C}_w \subseteq \mathcal{C}_s$ ). Thus these two notions are essentially equivalent.

**§3. Modal logic preliminaries.** Let  $L_M$  be the modal language corresponding to  $L_C$ , but including  $\Box$  and  $\Diamond$  among its primitives. Likewise let  $L_M^*$  and  $L_C^*$  correspond, i.e.,  $L_M^*$  has countably many more constants than  $L_M$ . We continue the  $\alpha, \beta, \gamma, \delta$  division and, following [4] add two more categories, necessities ( $\nu$ ) and possibles ( $\pi$ ). These, together with their instances, are given in the following charts.

$\nu$	$\nu_0$	$\pi$	$\pi_0$
$\Box X$	$X$	$\Diamond X$	$X$
$\sim \Diamond X$	$\sim X$	$\sim \Box X$	$\sim X$

We will have much to do with Kripke models, but the only ones we need to consider here have the constants of  $L_M^*$  as domain, each interpreted as naming itself. Consequently, to simplify notation we will suppress any mention of interpretation in our definition of Kripke models.

**DEFINITION.** By a *K model* (see [9]) we mean an ordered quadruple  $\langle G, R, \vDash, P \rangle$  where

- (1)  $G$  is a nonempty set (of possible worlds),
- (2)  $R$  is a relation on  $G$  (of relative possibility or accessibility),
- (3)  $P$  is a function from  $G$  to nonempty sets of constants of  $L_M^*$  satisfying the condition: for  $\Gamma, \Delta \in G$ , if  $\Gamma R \Delta$  then  $P(\Gamma) \subseteq P(\Delta)$  ( $P(\Gamma)$  is the set of “things” in the world  $\Gamma$ ),
- (4)  $\vDash$  is a relation between members of  $G$  and statements of  $L_M^*$  ( $\Gamma \vDash X$  means  $X$  is true in the world  $\Gamma$ ) satisfying, for each  $\Gamma \in G$ ,
  - (a)  $\Gamma \vDash \sim X \leftrightarrow \Gamma \not\vDash X$  (i.e., not  $\Gamma \vDash X$ ),
  - (b)  $\Gamma \vDash \alpha \leftrightarrow \Gamma \vDash \alpha_1$  and  $\Gamma \vDash \alpha_2$ ,
  - (c)  $\Gamma \vDash \beta \leftrightarrow \Gamma \vDash \beta_1$  or  $\Gamma \vDash \beta_2$ ,
  - (d)  $\Gamma \vDash \gamma \leftrightarrow \Gamma \vDash \gamma(c)$  for each  $c \in P(\Gamma)$ ,
  - (e)  $\Gamma \vDash \delta \leftrightarrow \Gamma \vDash \delta(c)$  for some  $c \in P(\Gamma)$ ,
  - (f)  $\Gamma \vDash \nu \leftrightarrow \Delta \vDash \nu_0$  for every  $\Delta \in G$  such that  $\Gamma R \Delta$ ,
  - (g)  $\Gamma \vDash \pi \leftrightarrow \Delta \vDash \pi_0$  for some  $\Delta \in G$  such that  $\Gamma R \Delta$ .

REMARK. In the above definition one each of (b) or (c), (d) or (e) and (f) or (g) is redundant.

DEFINITION.  $\langle G, R, \vDash, P \rangle$  is a *T model* if  $R$  is reflexive, and an *S4 model* if  $R$  is reflexive and transitive.

We note for future use that if  $G, R$  and  $P$  satisfy conditions (1), (2) and (3) and  $\vDash$  is specified for *atomic* statements of  $L_M^*$  then  $\vDash$  can be extended in one and only one way to a relation, again denoted by  $\vDash$ , so that  $\langle G, R, \vDash, P \rangle$  is a *K-model*. This may be shown by an induction on degree.

DEFINITION. Let  $S$  be a set of statements of  $L_M^*$ . We say  $S$  is *K-satisfiable* (*T satisfiable*, *S4 satisfiable*) if there is some *K model* (*T model*, *S4 model*)  $\langle G, R, \vDash, P \rangle$  and some  $\Gamma \in G$  with every constant of  $S$  in  $P(\Gamma)$  such that  $\Gamma \vDash X$  for every  $X \in S$ . We say  $X$  is *K valid* (*T valid*, *S4 valid*) if  $\{\sim X\}$  is not *K satisfiable* (*T satisfiable*, *S4 satisfiable*).

**§4. S4 consistency properties.** We use  $S_\nu$  for  $\{\nu \mid \nu \in S\}$ .

DEFINITION. Let  $\mathcal{C}$  be a collection of sets of statements of  $L_M^*$ . We call  $\mathcal{C}$  an *S4 consistency property* if it is a classical consistency property and in addition, for each  $S \in \mathcal{C}$ ,

- (6)  $\nu \in S \rightarrow S \cup \{\nu_0\} \in \mathcal{C}$ ,
- (7)  $\pi \in S \rightarrow S_\nu \cup \{\pi_0\} \in \mathcal{C}$ .

MODEL EXISTENCE THEOREM FOR S4. *Let  $S$  be any set of statements of  $L_M$ . If  $S$  belongs to some S4 consistency property,  $S$  is S4 satisfiable.*

The remainder of this section is devoted to a proof of the above theorem. First, we may define a notion of *strong S4 consistency property* by changing the appropriate clause in the above definition from “it is a classical consistency property” to “it is a strong classical consistency property.” Then, using the methods of §3 we may show the following:

LEMMA. *Any S4 consistency property may be extended to a strong S4 consistency property.*

LEMMA. *Let  $\mathcal{C}$  be a strong S4 consistency property. Let  $\mathcal{C}'$  be the result of enlarging  $\mathcal{C}$  by adding all unions of chains in  $\mathcal{C}$ . Then  $\mathcal{C}'$  is again a strong S4 consistency property.*

PROOF. The seven conditions of the definition must be verified for  $\mathcal{C}'$ . We check conditions (3) and (5') and leave the rest to the reader.

We deal with condition (5') first. Suppose  $S \in \mathcal{C}'$ ,  $\delta \in S$ , and  $c$  is a constant not appearing in  $S$ . Let  $S$  be  $\cup_i S_i$  where the  $S_i$  constitute a chain in  $\mathcal{C}$ .  $\delta \in S$ , so  $\delta$  belongs to each  $S_i$  from some point in the chain on. By discarding an initial segment of the chain we may suppose  $\delta$  belongs to each member of the chain. Since  $c$  does not appear in  $S$  it is not in any  $S_i$ . Then  $S_i \cup \{\delta(c)\} \in \mathcal{C}$  for each  $i$ . And  $S \cup \{\delta(c)\} = \cup_i [S_i \cup \{\delta(c)\}]$ , which is a chain union. Thus  $S \cup \{\delta(c)\} \in \mathcal{C}'$ .

To verify condition (3), let us suppose  $S = \cup_i S_i$  where the  $S_i$  constitute a chain in  $\mathcal{C}$ , and  $\beta \in S$ . As above we may assume  $\beta$  is in each  $S_i$ . Now for each  $i$ , either  $S_i \cup \{\beta_1\} \in \mathcal{C}$  or  $S_i \cup \{\beta_2\} \in \mathcal{C}$ . Let  $\mathcal{A}$  be the set of those  $S_i$  such that  $S_i \cup \{\beta_1\} \in \mathcal{C}$ , and let  $\mathcal{B}$  consist of those  $S_i$  such that  $S_i \cup \{\beta_2\} \in \mathcal{C}$ . Both  $\mathcal{A}$  and  $\mathcal{B}$  are chains, and either  $S = \cup \mathcal{A}$  or  $S = \cup \mathcal{B}$  depending on whether  $\mathcal{A}$  or  $\mathcal{B}$  is cofinal with

the chain. Say  $S = \bigcup \mathcal{A}$ . Then  $S \cup \{\beta_1\}$  is the union of the chain consisting of all  $S_i \cup \{\beta_1\}$  with  $S_i \in \mathcal{A}$ , thus  $S \cup \{\beta_1\} \in \mathcal{C}'$ . Similarly if  $S = \bigcup \mathcal{B}$ .

From these lemmas we immediately get the following:

**THEOREM.** *Any S4 consistency property may be enlarged to a strong S4 consistency property which is closed under chain unions.*

**DEFINITION.** Let  $D$  be a set of statements and  $C$  be a nonempty set of constants, including at least all those occurring in  $D$ . We say  $D$  is *downward saturated with respect to  $C$*  if

- (1) if  $A$  is atomic, not both  $A \in D$  and  $\sim A \in D$ ,
- (2)  $\alpha \in D \rightarrow \alpha_1 \in D$  and  $\alpha_2 \in D$ ,
- (3)  $\beta \in D \rightarrow \beta_1 \in D$  or  $\beta_2 \in D$ ,
- (4)  $\gamma \in D \rightarrow \gamma(c) \in D$  for each  $c \in C$ ,
- (5)  $\delta \in D \rightarrow \delta(c) \in D$  for some  $c \in C$ ,
- (6)  $\nu \in D \rightarrow \nu_0 \in D$ .

**KEY LEMMA.** *Let  $\mathcal{C}$  be a strong S4 consistency property which is closed under chain unions. Let  $S_0$  be a member of  $\mathcal{C}$  and  $C_0$  be the set of constants of  $S_0$ . Suppose  $\{c_1, c_2, c_3, \dots\}$  is a countable set of constants not in  $C_0$  and let  $C = C_0 \cup \{c_1, c_2, c_3, \dots\}$ . Then  $S_0$  has an extension  $S$  in  $\mathcal{C}$ , which is downward saturated with respect to  $C$ .*

**PROOF.** Let  $X_0, X_1, X_2, \dots$  be an ordering of the set of all statements involving only constants of  $C$ . We define a sequence,  $S_0, S_1, S_2, \dots$  of members of  $\mathcal{C}$ .

$S_0$  is given.

Suppose  $S_n$  has been defined, so that  $S_n \in \mathcal{C}$ , and only finitely many of  $\{c_1, c_2, c_3, \dots\}$  occur in  $S_n$ . We define an auxiliary finite sequence,  $s_n^0, s_n^1, s_n^2, \dots, s_n^n$  as follows.

Let  $s_n^0 = S_n$ .

Suppose  $s_n^k$  has been defined for some  $k < n$  so that  $s_n^k \in \mathcal{C}$  and only finitely many of  $\{c_1, c_2, c_3, \dots\}$  occur in  $s_n^k$ . Consider the statement  $X_k$ . If  $X_k$  does not belong to  $s_n^k$ , let  $s_n^{k+1} = s_n^k$ . If it does, we have several possibilities.

*Case 1.*  $X_k$  is atomic or negation of atomic. Let  $s_n^{k+1} = s_n^k$ .

*Case 2.*  $X_k$  is an  $\alpha$ . Let  $s_n^{k+1} = s_n^k \cup \{\alpha_1, \alpha_2\}$ .

*Case 3.*  $X_k$  is a  $\beta$ . Let  $s_n^{k+1} = s_n^k \cup \{\beta_1\}$  if that is in  $\mathcal{C}$ , otherwise let  $s_n^{k+1} = s_n^k \cup \{\beta_2\}$ .

*Case 4.*  $X_k$  is a  $\gamma$ . Since  $\mathcal{C}$  is closed under chain unions,  $s_n^k \cup \{\gamma(c) \mid c \in C_0\} \cup \{\gamma(c_1), \gamma(c_2), \dots, \gamma(c_n)\}$  belongs to  $\mathcal{C}$ . Let it be  $s_n^{k+1}$ .

*Case 5.*  $X_k$  is a  $\delta$ . Let  $c_i$  be the first of  $c_1, c_2, c_3, \dots$  not occurring in  $s_n^k$ . Let  $s_n^{k+1} = s_n^k \cup \{\delta(c_i)\}$ .

*Case 6.*  $X_k$  is a  $\nu$ . Let  $s_n^{k+1} = s_n^k \cup \{\nu_0\}$ .

*Case 7.*  $X_k$  is a  $\pi$ . Let  $s_n^{k+1} = s_n^k$ .

In each case,  $s_n^{k+1} \in \mathcal{C}$ , and  $s_n^{k+1}$  involves only finitely many of  $\{c_1, c_2, c_3, \dots\}$ .

Now, let  $S_{n+1} = s_n^n$ . Thus the sequence  $S_0, S_1, S_2, \dots$  has been defined. Each  $S_i \in \mathcal{C}$ , and clearly  $S_0 \subseteq S_1 \subseteq S_2 \subseteq \dots$ . Since  $\mathcal{C}$  is closed under chain unions,  $S = \bigcup_n S_n$  is in  $\mathcal{C}$ , and it is easy to see  $S$  is downward saturated with respect to  $C$ . This concludes the proof. Now, finally,

**PROOF OF MODEL EXISTENCE THEOREM FOR S4.** Let  $\mathcal{C}$  be a strong S4 consistency property which is closed under chain unions. Let  $S$  be a set of statements of  $L_M$ ,

and suppose  $S \in \mathcal{C}$ . We use  $\mathcal{C}$  to create an  $S4$  model  $\langle G, R, \vDash, P \rangle$  in which  $S$  is satisfiable.

First, partition the constants of  $L_M^*$  into countably many, countable, disjoint sets,  $C_1, C_2, C_3, \dots$ , so that all constants of  $L_M$  are in  $C_1$ . Let  $P_n = C_1 \cup C_2 \cup \dots \cup C_n$ . Now, let  $G$  consist of all ordered pairs,  $\langle \Gamma, P_n \rangle$ , where  $\Gamma$  is a member of  $\mathcal{C}$  which is downward saturated with respect to  $P_n$ . Let  $\langle \Gamma, P_n \rangle R \langle \Delta, P_k \rangle$  mean  $\Gamma_v \subseteq \Delta$  and  $P_n \subseteq P_k$ . Next, let  $P(\langle \Gamma, P_n \rangle) = P_n$ . Finally, if  $A$  is atomic, let  $\langle \Gamma, P_n \rangle \vDash A$  provided  $A \in \Gamma$ . Then  $\vDash$  can be extended so that  $\langle G, R, \vDash, P \rangle$  is an  $S4$  model. We claim, for any statement  $X$  of  $L_M^*$  and for any  $\langle \Gamma, P_n \rangle \in G$ ,

$$(*) \quad X \in \Gamma \rightarrow \langle \Gamma, P_n \rangle \vDash X.$$

This may be shown by an induction on the degree of  $X$ . It is immediate for atomic statements. Suppose now  $X$  is of degree  $k > 0$  and  $(*)$  is known for statements of lower degree. We have several cases.

If  $X$  is the negation of an atomic statement  $(*)$  is easily obtained using clause (1) of the definition of consistency property.

Suppose  $X$  is a  $\gamma$ . If  $X \in \Gamma$ , since  $\Gamma$  is downward saturated with respect to  $P_n$ ,  $\gamma(c) \in \Gamma$  for each  $c \in P_n$ . By the induction hypothesis,  $\langle \Gamma, P_n \rangle \vDash \gamma(c)$  for each  $c \in P_n$ , that is, for each  $c \in P(\langle \Gamma, P_n \rangle)$ . Then  $\langle \Gamma, P_n \rangle \vDash \gamma$ .

Suppose  $X$  is a  $\pi$ . If  $X \in \Gamma$ , since  $\Gamma \in \mathcal{C}$  and  $\mathcal{C}$  is a strong  $S4$  consistency property,  $\Gamma_v \cup \{\pi_0\} \in \mathcal{C}$ . By the Key Lemma,  $\Gamma_v \cup \{\pi_0\}$  may be extended to a set downward saturated with respect to  $P_{n+1}$ , call it  $\Delta$ . Then  $\langle \Delta, P_{n+1} \rangle \in G$ , and  $\langle \Gamma, P_n \rangle R \langle \Delta, P_{n+1} \rangle$ . Now  $\pi_0 \in \Delta$  so by the induction hypothesis,  $\langle \Delta, P_{n+1} \rangle \vDash \pi_0$ , hence  $\langle \Gamma, P_n \rangle \vDash \pi$ .

The other cases are left to the reader.

Now  $S \in \mathcal{C}$ .  $S$  may be extended to a set  $\Gamma$  downward saturated with respect to  $P_2$ .  $\langle \Gamma, P_2 \rangle \in G$ , and by  $(*)$ ,  $X \in S \rightarrow \langle \Gamma, P_2 \rangle \vDash X$ , hence  $S$  is  $S4$  satisfiable.

**§5. Applications.** The basic uses of the  $S4$  Model Existence Theorem are the same as those of the classical one. We summarize them.

I. *Completeness of axiom systems.* The completeness of any of the usual  $S4$  axiom systems follows once it has been shown that the collection of all sets which are consistent in the sense of the axiom system constitutes an  $S4$  consistency property.

II. *Completeness of tableau systems.* The classical tableau proof system using unsigned statements given on pp. 20 and 53 of [10] may be extended to a remarkably convenient proof system for first-order  $S4$  by the addition of two rules:

If  $\nu$  occurs on a branch,  $\nu_0$  may be added to the end of the branch.

If  $\pi$  occurs on a branch,  $\pi_0$  may be added to the end of the branch, but all statements on the branch which are not  $\nu$  statements (necessaries) must be crossed out first.

Schematically these may be given as

$$\frac{\nu}{\nu_0}, \quad \frac{\pi}{\pi_0} \quad (\text{but cross out all nonnecessaries}).$$

REMARK. In applying the second of these rules it may happen that we encounter

a statement occurrence common to several branches, and which must be crossed out on only one of these. As a practical device, delete it, then add fresh occurrences of it at the ends of the branches on which it is to remain undeleted.

The correctness of this tableau system may be established in the usual tableau way. Also, if we call a finite set of statements of  $L_M^*$  consistent provided no tableau for it closes, the collection of consistent sets constitutes an  $S4$  consistency property, and completeness follows.

III. *Completeness of Gentzen systems.* The completeness of Gentzen systems for  $S4$ , say those in the Appendix of [2], may be shown in the following way. Call a finite set of  $L_M^*$  statements  $\{X_1, X_2, \dots, X_n, \sim Y_1, \dots, \sim Y_k\}$  consistent if the sequent  $X_1, X_2, \dots, X_n \vdash Y_1, Y_2, \dots, Y_k$  is not provable. The collection of consistent sets is an  $S4$  consistency property and completeness follows.

IV. *Compactness Theorem.* If we let  $\mathcal{C}$  consist of those sets of statements of  $L_M^*$  every finite subset of which is  $S4$  satisfiable,  $\mathcal{C}$  is an  $S4$  consistency property. We thus have: If every finite subset of a set of  $L_M$  statements is  $S4$  satisfiable, so is the entire set.

V. *Lowenheim-Skolem Theorem.* For this paragraph only let us allow  $S4$  models involving uncountably many constants. Let  $\mathcal{C}$  be the collection of all subsets of  $L_M^*$  which are  $S4$  satisfiable, allowing these uncountable models.  $\mathcal{C}$  is an  $S4$  consistency property. Then the  $S4$  Model Existence Theorem and its proof give a Lowenheim-Skolem Theorem for  $S4$ .

VI. *Craig Interpolation Lemma.* If  $S$  is a finite set of statements, in the interests of simple notation we will sometimes use  $S$  to denote the conjunction of the members of  $S$ , grouped arbitrarily.

Let us say the statement  $X \supset Y$  has an *interpolant* if (1)  $\sim X$  is  $S4$  valid, or (2)  $Y$  is  $S4$  valid, or (3) there is a statement  $Z$  all of whose constant and relation symbols are common to  $X$  and  $Y$ , such that both  $X \supset Z$  and  $Z \supset Y$  are  $S4$  valid.

Now suppose we let  $\mathcal{C}$  consist of those finite sets  $S$  of  $L_M^*$  which can be partitioned into two disjoint subsets,  $S_1$  and  $S_2$ , so that  $S_1 \supset \sim S_2$  has no interpolant.  $\mathcal{C}$  is an  $S4$  consistency property. We leave the verification of most of the clauses to the reader and discuss only the  $\pi$  case.

Suppose  $S \in \mathcal{C}$  and  $\pi \in S$ . We show  $S_v \cup \{\pi_0\} \in \mathcal{C}$ . Since  $S \in \mathcal{C}$ ,  $S$  can be partitioned into  $S_1$  and  $S_2$  so that  $S_1 \supset \sim S_2$  has no interpolant.  $\pi$  is in one of  $S_1$  or  $S_2$ , let us say  $S_1$ ; the proof if  $\pi \in S_2$  is similar. To make the notation reflect that  $\pi$  is in  $S_1$ , we henceforth write  $S_1$  as  $S'_1 \cup \{\pi\}$  where  $\pi$  is not in  $S'_1$ . Thus we have that  $(S'_1 \wedge \pi) \supset \sim S_2$  has no interpolant.

Now, if  $S_v \cup \{\pi_0\}$  did not belong to  $\mathcal{C}$ ,  $(S'_{1v} \wedge \pi_0) \supset \sim S_{2v}$  would have an interpolant. There are three possibilities.

*Case 1.*  $\sim (S'_{1v} \wedge \pi_0)$  is  $S4$  valid. Then so is  $S'_{1v} \supset \sim \pi_0$ . Hence also  $\Box S'_{1v} \supset \Box \sim \pi_0$ . But, in  $S4$ ,  $\Box S'_{1v} \equiv S'_{1v}$ , and  $\Box \sim \pi_0 \equiv \sim \pi$ , so we have  $S'_{1v} \supset \sim \pi$ . From this we get  $S'_1 \supset \sim \pi$ , so finally,  $\sim (S'_1 \wedge \pi)$  is  $S4$  valid, contradicting the fact that  $(S'_1 \wedge \pi) \supset \sim S_2$  has no interpolant.

*Case 2.*  $\sim S_{2v}$  is  $S4$  valid. But  $S_2 \supset S_{2v}$ , so  $\sim S_2$  is also  $S4$  valid, again a contradiction.

*Case 3.* There is a statement  $I$  of all whose constant and relation symbols

are common to  $(S'_{1v} \wedge \pi_0)$  and  $\sim S_{2v}$ , such that both of the following are  $S_4$  valid:

- (1)  $(S'_{1v} \wedge \pi_0) \supset I$ .
- (2)  $I \supset \sim S_{2v}$ .

From (1) we obtain the  $S_4$  validity of

$$S'_{1v} \supset (\sim I \supset \sim \pi_0), \quad \Box S'_{1v} \supset (\Box \sim I \supset \Box \sim \pi_0), \\ (\Box S'_{1v} \wedge \sim \Box \sim \pi_0) \supset \sim \Box \sim I, \quad (S'_1 \wedge \pi) \supset \Diamond I.$$

From (2) we obtain the  $S_4$  validity of

$$S_{2v} \supset \sim I, \quad \Box S_{2v} \supset \Box \sim I, \quad S_2 \supset \Box \sim I, \quad \Diamond I \supset \sim S_2.$$

Thus  $(S'_1 \wedge \pi) \supset \sim S_2$  has an interpolant  $\Diamond I$ , again a contradiction. We conclude  $S_v \cup \{\pi_0\} \in \mathcal{C}$ .

Now the Craig Interpolation Lemma for  $S_4$  follows easily. Suppose  $X \supset Y$  is a statement of  $L_M$  which is  $S_4$  valid. Then  $\{X, \sim Y\}$  is not  $S_4$  satisfiable. By the Model Existence Theorem for  $S_4$ ,  $\{X, \sim Y\} \notin \mathcal{C}$ , so  $X \supset Y$  must have an interpolant.

**§6. The logics  $T$  and  $K$ .** I. By  $S_{v_0}$  we mean  $\{v_0 \mid v \in S\}$ . Now, by a  $T$  consistency property we mean a collection  $\mathcal{C}$  satisfying all the conditions for an  $S_4$  consistency property, except that (7) is replaced by

- (7') if  $\pi \in S \in \mathcal{C}$  then  $S_{v_0} \cup \{\pi_0\} \in \mathcal{C}$ .

The proof of the  $S_4$  Model Existence Theorem adapts to  $T$  simply by replacing  $S_v$  by  $S_{v_0}$  at appropriate points in the argument. Thus we have

**MODEL EXISTENCE THEOREM FOR  $T$ .** *If  $S$  is a set of statements of  $L_M$  which belongs to some  $T$  consistency property, then  $S$  is  $T$  satisfiable.*

The applications of this theorem are akin to those of §5 and are left to the reader. We remark, however, that the following constitutes a complete tableau system for  $T$ : Add to the classical system of [10] the following two rules:

If  $v$  occurs on a branch,  $v_0$  may be added to the end of the branch.

If  $\pi$  occurs on a branch,  $\pi_0$  may be added to the end of the branch, but all statements on the branch which are not  $v$  statements must be deleted, and any statement of the form  $v$  must be replaced by  $v_0$ .

II. By a  $K$  consistency property we mean a collection  $\mathcal{C}$  satisfying all the conditions for a  $T$  consistency property except possibly (6) if  $v \in S \in \mathcal{C}$  then  $S \cup \{v_0\} \in \mathcal{C}$ . Again the work of §4 easily adapts to show the following:

**MODEL EXISTENCE THEOREM FOR  $K$ .** *If  $S$  is a set of statements of  $L_M$  which belongs to some  $K$  consistency property, then  $S$  is  $K$  satisfiable.*

A tableau system for  $K$  results very simply if from the above tableau system for  $T$  we delete the rule: If  $v$  occurs on a branch,  $v_0$  may be added to the end of the branch. We leave the basic applications of this Model Existence Theorem to the reader.

**§7. Intuitionistic logic preliminaries.** Recall  $L_C$  is the classical language with  $\wedge, \vee, \sim, \supset, \forall$  and  $\exists$  primitive.  $L_C^*$  has countably many more constants. We will use Kripke intuitionistic logic models, but we will only consider those whose

domains consist of constants of  $L_C^*$  and so no notion of interpretation will be mentioned.

DEFINITION. By an *intuitionistic model* (see [9] and [3]) we mean an ordered quadruple  $\langle G, R, \vDash, P \rangle$  where

- (1)  $G$  is a nonempty set,
- (2)  $R$  is a transitive, reflexive relation on  $G$ ,
- (3)  $P$  is a function from  $G$  to nonempty sets of constants of  $L_C^*$  satisfying the condition  $\Gamma R \Delta \rightarrow P(\Gamma) \subseteq P(\Delta)$ ,
- (4)  $\vDash$  is a relation between members of  $G$  and statements of  $L_C^*$  satisfying, for each  $\Gamma \in G$ ,
  - (a)  $\Gamma \vDash A, \Gamma R \Delta \rightarrow \Delta \vDash A$ , for atomic,
  - (b)  $\Gamma \vDash (X \wedge Y) \leftrightarrow \Gamma \vDash X$  and  $\Gamma \vDash Y$ ,
  - (c)  $\Gamma \vDash (X \vee Y) \leftrightarrow \Gamma \vDash X$  or  $\Gamma \vDash Y$ ,
  - (d)  $\Gamma \vDash \sim X \leftrightarrow$  for each  $\Delta \in G$  such that  $\Gamma R \Delta$ , not  $\Delta \vDash X$ ,
  - (e)  $\Gamma \vDash (X \supset Y) \leftrightarrow$  for each  $\Delta \in G$  such that  $\Gamma R \Delta$ , if  $\Delta \vDash X$  then  $\Delta \vDash Y$ ,
  - (f)  $\Gamma \vDash (\forall x)A(x) \leftrightarrow$  for each  $\Delta \in G$  such that  $\Gamma R \Delta$ ,  $\Delta \vDash A(c)$  for each  $c \in P(\Delta)$ ,
  - (g)  $\Gamma \vDash (\exists x)A(x) \leftrightarrow \Gamma \vDash A(c)$  for some  $c \in P(\Gamma)$ .

As for modal logics,  $\vDash$  is completely determined if its behavior is known for atomic statements, but that may be arbitrarily specified.

*Important observation.* Condition (a) actually holds for all statements, not just atomic ones; proof is by induction on degree.

Note that  $\Gamma \vDash \sim X$  is not the same as  $\Gamma \# X$ . It thus becomes useful to introduce signed statements as in [3].

DEFINITION. By a *signed statement* we mean  $TX$  or  $FX$  where  $X$  is a statement of  $L_C^*$ . We use  $\Gamma \vDash TX$  as synonymous with  $\Gamma \vDash X$ , and  $\Gamma \vDash FX$  with  $\Gamma \# X$ .

If  $S$  is a set of signed statements we say  $S$  is *intuitionistically satisfiable* if there is some intuitionistic model,  $\langle G, R, \vDash, P \rangle$ , and some  $\Gamma \in G$  with every constant of  $S$  in  $P(\Gamma)$ , such that  $\Gamma \vDash Z$  for each  $Z \in S$ . (This was called realizability in [3].) We say a set  $S$  of unsigned statements is intuitionistically satisfiable if  $\{TX \mid X \in S\}$  is intuitionistically satisfiable. Finally we say  $X$  is intuitionistically valid if  $\{FX\}$  is not intuitionistically satisfiable.

For some of our work it is convenient to continue using uniform notation, but it must be modified to apply to *signed* statements. The following charts (see [10]) do this.

$\alpha$	$\alpha_1$	$\alpha_2$	$\beta$	$\beta_1$	$\beta_2$
$T(X \wedge Y)$	$TX$	$TY$	$T(X \vee Y)$	$TX$	$TY$
$F(X \vee Y)$	$FX$	$FY$	$F(X \wedge Y)$	$FX$	$FY$
(*) $F(X \supset Y)$	$TX$	$FY$	$T(X \supset Y)$	$FX$	$TY$
(*) $F \sim X$	$TX$	$TX$	$T \sim X$	$FX$	$FX$
$\gamma$	$\gamma(c)$		$\delta$	$\delta(c)$	
$T(\forall x)A(x)$	$TA(c)$		$T(\exists x)A(x)$	$TA(c)$	
$F(\exists x)A(x)$	$FA(c)$		(*) $F(\forall x)A(x)$	$FA(c)$	

The starred signed statements,  $F \sim X$ ,  $F(X \supset Y)$  and  $F(\forall x)A(x)$  are called *special*. All other signed statements are *regular*.

**§8. Intuitionistic logic consistency properties.** Unlike the logics discussed so far, there is no unique “natural” notion of consistency property for intuitionistic logic, rather there are several, since stricter or more liberal versions of some of the closure conditions are possible. The two extremes are what we deal with; arising from work of Beth and Gentzen.

We use  $S_T$  for  $\{TX \mid TX \in S\}$ .

**DEFINITION.** By a *Beth (intuitionistic) consistency property* we mean a collection  $\mathcal{C}$  of sets of signed statements of  $L_c^*$  such that, for each  $S \in \mathcal{C}$ ,

- (1) if  $A$  is atomic, not both  $TA \in S$  and  $FA \in S$ ,
- (2a) if  $\alpha$  is regular,  $\alpha \in S \rightarrow S \cup \{\alpha_1, \alpha_2\} \in \mathcal{C}$ ,
- (2b) if  $\alpha$  is special,  $\alpha \in S \rightarrow S_T \cup \{\alpha_1, \alpha_2\} \in \mathcal{C}$ ,
- (3)  $\beta \in S \rightarrow S \cup \{\beta_1\} \in \mathcal{C}$  or  $S \cup \{\beta_2\} \in \mathcal{C}$ ,
- (4)  $\gamma \in S \rightarrow S \cup \{\gamma(c)\} \in \mathcal{C}$  for each constant  $c$ ,
- (5a) if  $\delta$  is regular,  $\delta \in S \rightarrow S \cup \{\delta(c)\} \in \mathcal{C}$  for some constant  $c$ ,
- (5b) if  $\delta$  is special,  $\delta \in S \rightarrow S_T \cup \{\delta(c)\} \in \mathcal{C}$  for some constant  $c$ .

It is possible to adapt the work of §4 along the lines of §§3, 4 and 5 of Chapter 5 of [3] to produce a direct proof of an intuitionistic model existence theorem for this notion of consistency property. We do not take this route however. Instead we introduce a second notion of intuitionistic consistency property and work with it directly, and thus with the above indirectly. For this, uniform notation is no longer useful.

**DEFINITION.** Let  $\mathcal{C}$  be a collection of sets of signed statements of  $L_c^*$ . We call  $\mathcal{C}$  a *Gentzen (intuitionistic) consistency property* if, for each  $S \in \mathcal{C}$ ,

- (1) if  $A$  is atomic, not both  $TA \in S$  and  $FA \in S$ ,
- (2)  $T(X \wedge Y) \in S \rightarrow S \cup \{TX, TY\} \in \mathcal{C}$ ,
- (3)  $F(X \wedge Y) \in S \rightarrow S_T \cup \{FX\} \in \mathcal{C}$  or  $S_T \cup \{FY\} \in \mathcal{C}$ ,
- (4)  $T(X \vee Y) \in S \rightarrow S \cup \{TX\} \in \mathcal{C}$  or  $S \cup \{TY\} \in \mathcal{C}$ ,
- (5)  $F(X \vee Y) \in S \rightarrow S_T \cup \{FX\} \in \mathcal{C}$  and  $S_T \cup \{FY\} \in \mathcal{C}$ ,
- (6)  $T \sim X \in S \rightarrow S_T \cup \{FX\} \in \mathcal{C}$ ,
- (7)  $F \sim X \in S \rightarrow S_T \cup \{TX\} \in \mathcal{C}$ ,
- (8)  $T(X \supset Y) \in S \rightarrow S_T \cup \{FX\} \in \mathcal{C}$  or  $S \cup \{TY\} \in \mathcal{C}$ ,
- (9)  $F(X \supset Y) \in S \rightarrow S_T \cup \{TX, FY\} \in \mathcal{C}$ ,
- (10)  $T(\forall x)A(x) \in S \rightarrow S \cup \{TA(c)\} \in \mathcal{C}$  for every constant  $c$ ,
- (11)  $F(\forall x)A(x) \in S \rightarrow S_T \cup \{FA(c)\} \in \mathcal{C}$  for some constant  $c$ ,
- (12)  $T(\exists x)A(x) \in S \rightarrow S \cup \{TA(c)\} \in \mathcal{C}$  for some constant  $c$ ,
- (13)  $F(\exists x)A(x) \in S \rightarrow S_T \cup \{FA(c)\} \in \mathcal{C}$  for every constant  $c$ .

**LEMMA.** Let  $\mathcal{B}$  be a Beth consistency property. Let  $\mathcal{G}$  consist of those sets  $S$  of signed statements such that for some  $S^* \in \mathcal{B}$ ,  $S_T = S_T^*$  and  $S_F \subseteq S_F^*$  ( $S_F = \{FX \mid FX \in S\}$ ). Then  $\mathcal{G}$  is a Gentzen consistency property.

**COROLLARY.** Every Beth consistency property may be extended to a Gentzen consistency property.

**LEMMA.** A Gentzen consistency property may be extended to a strong Gentzen

consistency property (that is, one meeting the above conditions, but with (11) and (12) replaced by

- (11')  $F(\forall x)A(x) \in S \rightarrow S_T \cup \{FA(c)\} \in \mathcal{C}$  for each constant  $c$  not appearing in  $S$ ,
- (12')  $T(\exists x)A(x) \in S \rightarrow S \cup \{TA(c)\} \in \mathcal{C}$  for each constant  $c$  not appearing in  $S$ ).

LEMMA. Any strong Gentzen consistency property can be extended to a strong Gentzen consistency property closed under chain unions.

DEFINITION. Let  $D \in \mathcal{C}$  (a Gentzen consistency property) and let  $C$  be a non-empty set of constants, including at least all those occurring in  $D$ . We say  $D$  is *T-saturated with respect to C* if

- (1) if  $A$  is atomic, not both  $TA \in D$  and  $FA \in D$ ,
- (2)  $T(X \wedge Y) \in D \rightarrow TX \in D$  and  $TY \in D$ ,
- (3)  $T(X \vee Y) \in D \rightarrow TX \in D$  or  $TY \in D$ ,
- (4)  $T(X \supset Y) \in D$  and  $D \cup \{TY\} \in \mathcal{C} \rightarrow TY \in D$ ,
- (5)  $T(\forall x)A(x) \in D \rightarrow TA(c) \in D$  for each  $c \in C$ ,
- (6)  $T(\exists x)A(x) \in D \rightarrow TA(c) \in D$  for some  $c \in C$ .

KEY LEMMA. Let  $\mathcal{C}$  be a strong Gentzen consistency property which is closed under chain unions. Let  $S_0 \in \mathcal{C}$  and let  $C_0$  be the set of constants of  $S_0$ . Suppose  $\{c_1, c_2, c_3, \dots\}$  is a countable set of constants none of which appear in  $C_0$ , and let  $C = C_0 \cup \{c_1, c_2, c_3, \dots\}$ . Then  $S_0$  has an extension  $S$  in  $\mathcal{C}$  which is *T-saturated with respect to C*.

MODEL EXISTENCE THEOREM FOR INTUITIONISTIC LOGIC. Let  $S$  be a set of signed statements of  $L_C$ . If  $S$  belongs to either a Beth or a Gentzen consistency property,  $S$  is intuitionistically satisfiable.

PROOF. By the preceding, it suffices to show this for the special case  $S \in \mathcal{C}$  where  $\mathcal{C}$  is a strong Gentzen consistency property closed under chain unions. Partition the constants of  $L_C^*$  into countably many countable, disjoint sets,  $C_1, C_2, C_3, \dots$  so that all constants of  $L_C$  are in  $C_1$ . Set  $P_n = C_1 \cup C_2 \cup \dots \cup C_n$ . Let  $G$  consist of all ordered pairs  $\langle \Gamma, P_n \rangle$  where  $\Gamma$  is a member of  $\mathcal{C}$  which is *T-saturated with respect to  $P_n$* . Let  $\langle \Gamma, P_n \rangle R \langle \Delta, P_k \rangle$  mean  $\Gamma_T \subseteq \Delta$  and  $P_n \subseteq P_k$ . Next, let  $P(\langle \Gamma, P_n \rangle) = P_n$ . Finally, if  $A$  is atomic, let  $\langle \Gamma, P_n \rangle \vDash A$  if  $TA \in \Gamma$ . This determines an intuitionistic model  $\langle G, R, \vDash, P \rangle$ . We claim, for each  $\langle \Gamma, P_n \rangle \in G$  and each statement  $X$  of  $L_C^*$ ,

$$(*) \quad \begin{aligned} TX \in \Gamma &\rightarrow \langle \Gamma, P_n \rangle \vDash X, \\ FX \in \Gamma &\rightarrow \langle \Gamma, P_n \rangle \not\vDash X. \end{aligned}$$

(\*) is shown by induction on the degree of  $X$ .

Case 1.  $X$  is atomic. Then (\*) is valid by definition of  $\vDash$  and clause (1) of the definition of Gentzen consistency property.

Now suppose (\*) is known for statements of degree less than  $k > 0$ , and  $X$  is of degree  $k$ . Suppose first that  $X$  is  $(A \wedge B)$ .

Case 2.  $TX \in \Gamma$ , i.e.,  $T(A \wedge B) \in \Gamma$ . Since  $\Gamma$  is *T-saturated with respect to  $P_n$* ,  $TA \in \Gamma$  and  $TB \in \Gamma$ . By the induction hypothesis,  $\langle \Gamma, P_n \rangle \vDash A$  and  $\langle \Gamma, P_n \rangle \vDash B$ , so  $\langle \Gamma, P_n \rangle \vDash (A \wedge B)$ .

Case 3.  $FX \in \Gamma$ , i.e.,  $F(A \wedge B) \in \Gamma$ . Then either  $\Gamma_T \cup \{FA\} \in \mathcal{C}$  or  $\Gamma_T \cup \{FB\} \in \mathcal{C}$ , say the former. By the Key Lemma,  $\Gamma_T \cup \{FA\}$  may be extended to some  $\Delta \in \mathcal{C}$  which is *T-saturated with respect to  $P_{n+1}$* .  $\langle \Delta, P_{n+1} \rangle \in G$ , and  $\langle \Gamma, P_n \rangle R \langle \Delta, P_{n+1} \rangle$ .

Moreover, by the induction hypothesis, since  $FA \in \Delta$ ,  $\langle \Delta, P_{n+1} \rangle \not\# A$ . It follows that  $\langle \Gamma, P_n \rangle \not\# A$ , so  $\langle \Gamma, P_n \rangle \vDash (A \wedge B)$ .

We leave the remaining cases to the reader.

We conclude this section with the remark that this work can be adapted to minimal logic by defining  $\sim X$  as  $X \supset f$ , and then postulating no special conditions on  $f$ . We do not carry out the details.

**§9. Basic applications.** I. *Completeness of axiom systems.* If we choose some standard axiom system for intuitionistic logic and let  $\mathcal{C}$  consist of those finite sets  $\{TX_1, \dots, TX_n, FY_1, \dots, FY_k\}$  such that  $(t \wedge X_1 \wedge \dots \wedge X_n) \supset (f \vee Y_1 \vee \dots \vee Y_k)$  is not provable (where  $t$  is a constant-free theorem and  $f$  is its negation),  $\mathcal{C}$  is a Beth consistency property. Now completeness follows.

II. *Completeness of tableau systems.* The classical tableau system of [10] using signed statements needs only a few simple changes to become an intuitionistic system. The rules

$$\frac{\beta}{\beta_1 \mid \beta_2}, \quad \frac{\gamma}{\gamma(c)} \text{ for any constant } c$$

remain the same. To the  $\alpha$  and  $\delta$  rules a proviso is added

$$\frac{\alpha}{\alpha_1}, \quad \frac{\delta}{\delta(c)} \text{ for any unused } c, \text{ proviso}$$

$\alpha_2$  proviso

where the proviso is: If the signed statement to which the rule is applied is *special*, all  $F$ -signed statements on the branch must be crossed out before the result of the rule is added to the end of the branch. (An intuitive explanation of this proviso is given in [3].)

If we let  $\mathcal{C}$  consist of those finite sets  $S$  of signed statements such that no tableau for  $S$  closes,  $\mathcal{C}$  is a Beth consistency property, and completeness follows easily. This is the basic tableau system of [3]. The alternate tableau system of §4 of Chapter 6 has a similar completeness proof, using Gentzen consistency properties. We leave this to the reader.

III. *Completeness of Gentzen systems.* Let  $\mathcal{C}$  consist of those finite sets  $\{TX_1, \dots, TX_n, FY\}$  such that  $X_1, \dots, X_n \rightarrow Y$  is not provable in the Gentzen system  $LJ$  of [7]. Then  $\mathcal{C}$  is a Gentzen consistency property, and completeness of  $LJ$  follows.

Likewise let  $\mathcal{C}$  consist of those finite sets  $\{TX_1, \dots, TX_n, FY_1, \dots, FY_k\}$  such that  $X_1, \dots, X_n \vdash Y_1, \dots, Y_k$  is not provable in Beth's system given on p. 449 of [1].  $\mathcal{C}$  is a Beth consistency property, and again completeness follows.

IV. *Compactness Theorem.* Let  $\mathcal{C}$  consist of those sets of signed statements of  $L_C^*$  every finite subset of which is intuitionistically satisfiable.  $\mathcal{C}$  is a Beth consistency property. Thus

(1) if  $S$  is a set of signed statements of  $L_C$ , every finite subset of which is intuitionistically satisfiable, then  $S$  itself is intuitionistically satisfiable.

And also the weaker result

(2) If  $S$  is a set of (unsigned) statements of  $L_C$ , and if every finite subset of  $S$  is intuitionistically satisfiable,  $S$  itself is intuitionistically satisfiable.

V. *Lowenheim-Skolem Theorem.* As in paragraph IV, signed and unsigned versions of a Lowenheim-Skolem Theorem for intuitionistic logic are provable, using the method of V of §5.

VI. *Craig Interpolation Lemma.* Let  $S_1$  and  $S_2$  be disjoint sets of signed statements of  $L_C^*$  with no  $F$ -signed statements in  $S_1$  and at most one in  $S_2$ . We say  $\langle S_1, S_2 \rangle$  has an interpolant if (1)  $S_1$  is intuitionistically unsatisfiable, or (2)  $S_2$  is intuitionistically unsatisfiable, or (3) there is a statement  $X$  all of whose constants and relation symbols are common to  $S_1$  and  $S_2$  such that both  $S_1 \cup \{FX\}$  and  $S_2 \cup \{TX\}$  are intuitionistically unsatisfiable. Let  $\mathcal{C}$  consist of all finite sets  $S$  which can be partitioned into disjoint subsets  $S_1$  and  $S_2$  so that  $\langle S_1, S_2 \rangle$  has no interpolant. Then  $\mathcal{C}$  is a Gentzen consistency property. We verify a few cases.

*Case 1.*  $F(A \vee B) \in S \in \mathcal{C}$ . Suppose  $S_T \cup \{FA\} \notin \mathcal{C}$ . We derive a contradiction. Since  $S \in \mathcal{C}$ ,  $S$  can be partitioned into  $S_1$  and  $S_2$  (with  $F(A \vee B) \in S_2$ ) so that  $\langle S_1, S_2 \rangle$  has no interpolant. Since  $S_T \cup \{FA\} \notin \mathcal{C}$ ,  $\langle S_{1T}, S_{2T} \cup \{FA\} \rangle$  has an interpolant. If  $S_{1T}$  is unsatisfiable, so is  $S_1$ . Likewise if  $S_{2T} \cup \{FA\}$  is unsatisfiable, so is  $S_2$ . Finally, if  $X$  is an interpolant for  $\langle S_{1T}, S_{2T} \cup \{FA\} \rangle$  it also is for  $\langle S_1, S_2 \rangle$ .

*Case 2.*  $T \sim A \in S \in \mathcal{C}$ . Suppose  $S_T \cup \{FA\} \notin \mathcal{C}$ . Since  $S \in \mathcal{C}$ ,  $S$  can be partitioned into  $S_1$  and  $S_2$  so that  $\langle S_1, S_2 \rangle$  has no interpolant. Now we have two sub-cases.

2a.  $T \sim A \in S_1$ . Since  $S_T \cup \{FA\} \notin \mathcal{C}$ ,  $\langle S_{2T}, S_{1T} \cup \{FA\} \rangle$  has an interpolant. If  $S_{2T}$  is unsatisfiable, so is  $S_2$ . If  $S_{1T} \cup \{FA\}$  is unsatisfiable so is  $S_1$ . Finally, if  $X$  is an interpolant, then  $\sim X$  is an interpolant for  $\langle S_1, S_2 \rangle$ . In any event,  $\langle S_1, S_2 \rangle$  has an interpolant.

2b.  $T \sim A \in S_2$ . Again since  $S_T \cup \{FA\} \notin \mathcal{C}$ ,  $\langle S_{1T}, S_{2T} \cup \{FA\} \rangle$  has an interpolant. If  $S_{1T}$  is unsatisfiable, so is  $S_1$ . If  $S_{2T} \cup \{FA\}$  is unsatisfiable, so is  $S_2$ . Finally, if  $X$  is an interpolant, then  $X$  is again an interpolant for  $\langle S_1, S_2 \rangle$ .

The remaining cases are left to the reader.

**§10. Embedding theorems.** There are several interesting translations between  $S_4$ ,  $S_5$ , classical and intuitionistic logics. In this section we show that classical consistency properties may be used to give simple model-theoretic proofs of several of them.

**THEOREM.** *If  $A$  is a statement of  $L_C$  without universal quantifiers,  $A$  is classically valid if and only if  $\sim \sim A$  is intuitionistically valid.*

**PROOF.** If  $\sim \sim A$  is intuitionistically valid,  $A$  is valid classically. This may be shown model-theoretically by observing that the one-world intuitionistic models are (isomorphically) the classical models.

If  $S$  is a set of statements of  $L_C^*$ , let  $\sim \sim S = \{\sim \sim X \mid X \in S\}$ . Let  $\mathcal{C}$  consist of all sets  $S$  of statements of  $L_C^*$  in which no universal quantifiers occur, and such that  $\sim \sim S$  is intuitionistically satisfiable. We claim  $\mathcal{C}$  is a classical consistency property.

Suppose  $\sim(X \wedge Y) \in S \in \mathcal{C}$ . Then  $\sim \sim S$  is intuitionistically satisfiable, say in

the world  $\Gamma$  of the intuitionistic model  $\langle G, R, \vDash, P \rangle$ . Then  $\Gamma \vDash \sim \sim \sim (X \wedge Y)$ . But  $\sim \sim \sim (X \wedge Y) \supset \sim \sim (\sim \sim \sim X \vee \sim \sim \sim Y)$  is intuitionistically valid, so  $\Gamma \vDash \sim \sim (\sim \sim \sim X \vee \sim \sim \sim Y)$ . It follows that there is some  $\Delta \in G$  with  $\Gamma R \Delta$  such that  $\Delta \vDash (\sim \sim \sim X \vee \sim \sim \sim Y)$ , so  $\Delta \vDash \sim \sim \sim X$  or  $\Delta \vDash \sim \sim \sim Y$ . But since all members of  $\sim \sim S$  hold at  $\Gamma$ , they also hold at  $\Delta$ . Thus either  $\sim \sim S \cup \{\sim \sim \sim X\}$  or  $\sim \sim S \cup \{\sim \sim \sim Y\}$  is intuitionistically satisfiable, so  $S \cup \sim \{X\} \in \mathcal{C}$  or  $S \cup \sim \{Y\} \in \mathcal{C}$ .

Suppose  $\sim(\exists x)A(x) \in S \in \mathcal{C}$  and  $c$  is some constant. Say  $\sim \sim S$  is satisfied at  $\Gamma$  in the intuitionistic model  $\langle G, R, \vDash, P \rangle$ . We may suppose  $c \in P(\Delta)$  for some  $\Delta \in G$  with  $\Gamma R \Delta$ , for if it does not we may choose some  $d \in P(\Gamma)$  and add  $c$  to  $P(\Delta)$  for each  $\Delta \in G$  with  $\Gamma R \Delta$ , specifying that  $c$  is to “behave like”  $d$ . Thus, suppose  $\Gamma R \Delta$  and  $c \in P(\Delta)$ . All members of  $\sim \sim S$  are true at  $\Delta$ , so  $\Delta \vDash \sim \sim \sim (\exists x)A(x)$ . It follows that  $\Delta \vDash \sim \sim \sim A(c)$ . Then  $\sim \sim S \cup \{\sim \sim \sim A(c)\}$  is intuitionistically satisfiable, so  $S \cup \sim \{A(c)\} \in \mathcal{C}$ .

We leave the remaining cases to the reader. Now, if  $\sim \sim A$  is not intuitionistically valid,  $\{\sim A\}$  must be intuitionistically satisfiable, hence so is  $\{\sim \sim A\}$ . Then  $\{\sim A\} \in \mathcal{C}$ ,  $\sim A$  is classically satisfiable,  $A$  is not classically valid.

LEMMA. *Let  $X$  be a statement of  $L_C^*$ . If each atomic subformula of  $X$  is immediately preceded by  $\sim \sim$ , and if  $X$  has no occurrences of  $\vee$  or  $\exists$ , then  $X$  is stable; that is  $\sim \sim X \supset X$  is intuitionistically valid.*

PROOF. By induction on the degree of  $X$ .

DEFINITION. Let  $X'$  be the result of inserting  $\sim \sim$  before every atomic subformula of  $X$ .

THEOREM. *If  $A$  is a statement of  $L_C$  without any occurrences of  $\vee$  or  $\exists$ ,  $A$  is classically valid if and only if  $A'$  is intuitionistically valid.*

PROOF. If  $A'$  is intuitionistically valid,  $A$  is valid classically, again by taking one-world models into account.

For sets  $S$  of  $L_C^*$  statements, define  $S' = \{X' \mid X \in S\}$ . Let  $\mathcal{C}$  consist of all sets  $S$  of  $L_C^*$  statements in which  $\vee$  and  $\exists$  do not occur, such that  $S'$  is intuitionistically satisfiable. We claim  $\mathcal{C}$  is a classical consistency property.

Suppose  $\sim(\forall x)A(x) \in S \in \mathcal{C}$ .  $S'$  is intuitionistically satisfiable, say at the world  $\Gamma$  in the intuitionistic model  $\langle G, R, \vDash, P \rangle$ . Now  $\sim \sim (\exists y)[\sim(\forall x)\sim \sim F(x) \supset \sim F(y)]$  is intuitionistically valid, so, since  $A'$  is stable,  $\sim \sim (\exists y)[\sim(\forall x)A'(x) \supset \sim A'(y)]$  is also intuitionistically valid, hence true at  $\Gamma$ . Then, for some  $\Delta \in G$  with  $\Gamma R \Delta$ ,  $\Delta \vDash (\exists y)[\sim(\forall x)A'(x) \supset \sim A'(y)]$ , so for some  $c \in P(\Delta)$ ,  $\Delta \vDash \sim(\forall x)A'(x) \supset \sim A'(c)$ . But all members of  $S'$ , being true at  $\Gamma$  are also true at  $\Delta$ , hence  $\Delta \vDash \sim(\forall x)A'(x)$ . Then  $\Delta \vDash \sim A'(c)$ ,  $S' \cup \{\sim A'(c)\}$  is intuitionistically satisfiable,  $S \cup \sim \{A(c)\} \in \mathcal{C}$ .

We leave the other cases to the reader. Now, if  $A'$  is not intuitionistically valid, neither is  $\sim \sim A'$ , since  $A'$  is stable. Then  $\{\sim A'\}$  must be intuitionistically satisfiable, so, as above,  $A$  is not classically valid.

Let  $X^\circ$  be the result of inserting  $\square$  before every subformula of  $X$ .

THEOREM. *For any statement  $A$  of  $L_C$ ,  $A$  is classically valid if and only if  $A^\circ$  is S5 valid (S5 Kripke models are those in which  $R$  is an equivalence relation).*

PROOF. If  $A^\circ$  is S5 valid,  $A$  is classically valid. This follows since the one-world S5 models are essentially the classical models.

Let  $S^\circ = \{X^\circ \mid X \in S\}$ . Let  $\mathcal{C}$  consist of all  $L_C^*$  sets  $S$  such that  $S^\circ$  is S5 satisfiable.

$\mathcal{C}$  is a classical consistency property. Now, if  $A^\circ$  is not  $S5$  valid,  $(\sim A)^\circ$  must be  $S5$  satisfiable, and the theorem follows.

Let  $X^*$  be the result of inserting  $\Box\Diamond$  before every subformula of  $X$ .

**THEOREM.** *For any statement  $A$  of  $L_C$ ,  $A$  is classically valid if and only if  $A^*$  is  $S4$  valid.*

The proof of this is basically the same as that of the previous theorem. Demonstrating that  $\mathcal{C} = \{S \mid S^*$  is  $S4$  satisfiable $\}$  is a classical consistency property is more difficult. We observe that

$$[\sim(X \wedge Y)]^* \supset \Diamond[(\sim X)^* \vee (\sim Y)^*] \quad \text{and} \quad \Diamond(\exists x)\{[(\exists y)A(y)]^* \supset [A(x)]^*\}$$

are  $S4$  valid. These will assist in such a demonstration, but we leave details to the reader.

We conclude with the remark that it is possible to use  $S4$  and intuitionistic consistency properties to show the  $X$  to  $X^\circ$  translation is an embedding of intuitionistic logic in  $S4$ , but it is much simpler to use Kripke  $S4$  and intuitionistic logic models directly to obtain this result.

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HERBERT H. LEHMAN COLLEGE (CUNY)  
BRONX, NEW YORK 10468