Non-classical logics and the independence results of set theory

by

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In the 1960's many famous independence results in set theory were first established using a newly created technique called *forcing*. Once this technique was discovered very quickly other, quite different, techniques for achieving the same results were found, techniques which are as interesting in themselves as the uses to which they may be put. In this paper we discuss some of these techniques, principally those based on using the well known non-classical logics S4 and intuitionistic logic. We try to show how such a role for non-classical logics is a natural development, one which makes such logics into things of use rather than the objects of curiosity they have traditionally been to the classically oriented mathematician. Finally we discuss briefly the notions of forcing and Boolean valued models and the interrelations of all these approaches to the problem.

A time honored method for proving the consistency of theory B relative to theory A has been to find within an intuitive model for theory A a model for theory B. Having done so, the work generally may be converted into a finitistic, proof-theoretic argument that the consistency of theory A implies the consistency of theory B. Probably the classic example of such a relative consistency proof is the familiar one for a (two dimensional) non-Euclidean geometry relative to (three dimensional) Euclidean geometry. If one fixes a sphere in Euclidean space, and interprets the non-Euclidean term 'line' to refer to a great circle on this sphere, a model for the non-Euclidean axioms is produced. To turn this argument into a finitistic relative consistency proof now is simple. We define the 'translate' X^T of a statement X of non-Euclidean geometry as follows. In X replace each reference to 'line' with a reference to 'great circle on a sphere G'. Then it may be shown that if X is any axiom of the non-Euclidean geometry, X^T is a theorem of Euclidean geometry. It follows that the translate of each non-Euclidean theorem is a Euclidean theorem. In particular, if a contradiction were derivable in non-Euclidean geometry, its translate, still a contradiction, would be a Euclidean theorem.

Another classic example of such a relative consistency proof is that for Euclidean geometry relative to real analysis. Here the 'translate' is defined using Cartesian coordinates.

Gödel's proof of the consistency of the generalized continuum hypothesis (GCH) is of the above pattern [6, 7]. He produces a model for the axioms of set theory together with GCH within an intuitive model for the axioms of set theory alone. As above, this produces a finitistic relative consistency proof. In this case the model is that of the 'constructible' sets, those sets which must occur in every model for the axioms of set theory which contains all the ordinal numbers. The translation procedure corresponding to those mentioned above is, informally, replace every reference in X to 'set' by a reference to 'constructible set'. Call the resulting statement X^L. Then, if X is an axiom of set theory, or if X is the GCH, X^L is a theorem of set theory. Once again, if a contradiction followed from the axioms of set theory together with GCH, the translate of that contradiction would be a theorem of set theory alone, which would thus be inconsistent.

Now, in order that we may carry out the translation described in the above paragraph, the notion of constructible set must be definable without reference to notions like "in every model". This can be done. There is a formula of first order logic, $L(x)$, having one free variable, with $\in$ as the sole predicate symbol, which, in each model for the axioms of set theory, is true of precisely the constructible sets. Thus, the translation procedure may be more properly specified as follows. Let X be a first order logic statement having $\in$ as its sole relation symbol. By X^L (called "X relativised to L") we mean the result of replacing, in X, each subformula of the form $(\forall x)P(x)$ by $(\forall x)[L(x) \supset P(x)]$ and of the form $(\exists x)P(x)$ by $(\exists x)[L(x) \wedge P(x)]$. Informally, we say the formula

$L(x)$ defines the model of constructible sets within any model for set theory. Such models, those definable within set theory models by formulas, are called *inner models*. Relative consistency proofs in set theory using inner models are intuitively natural things, and are clearly akin to the relative consistency proofs of geometry discussed above.

Unfortunately, there are inherent limitations on the method of proving relative consistency using inner models. It can be shown, for example, that this method is inadequate to establish the independence of GCH [12]. That is, there is no formula $F(x)$ in the language of set theory, having the property that X^F (X relativised to F) is provable in set theory where X is any axiom of set theory or is the *negation* of GCH. To show independence, then, something else must be tried; possibly a natural generalization of the method of inner models.

There are many interesting first order logics besides classical logic: intuitionistic logic, modal logics, many-valued logics; and for each of these adequate and intuitive model theories have been developed. The Shepherdson result about inner models says there is no *classical* logic inner model in which the axioms of set theory are true but the GCH is false. It does not say anything about the possibility of there being a formula $F(x)$ which defines, in each model for set theory, a (Kripke) *intuitionistic* logic model in which all the axioms of set theory hold, as well as the negation of GCH. Thus it does not rule out an inner model sort of proof of a result like: if classical set theory is consistent then the GCH can not be deduced from the axioms of set theory using intuitionistic logic. (I do not know if such a proof can be carried out.) Of course, such a result, if established, would be only a curiosity. What we want is an independence result in classical logic.

There are, however, several well-known connections between classical and intuitionistic logic which are of interest here. One such is a result of Gödel [4] which says: if S is a set of statements none of which involve $\vee$ or $\exists$, and X is a statement likewise not involving $\vee$ or $\exists$, then

$$S \vdash_C X \text{ if and only if } S \vdash_I X.$$

(We are using $\vdash_C$ and $\vdash_I$ for classical and intuitionistic deducibility respectively.) Now, teaming this up with the discussion above, we have the following more promising possibility. If we could produce an inner intuitionistic model in which all the axioms of set theory and the negation of GCH held, but expressed in (classically equivalent) forms which do not involve $\vee$ or $\exists$, then we would have the *classical* independence of GCH. (Again I do not know if this can be done.)

Another possibility is given by the following from [9]. If S and X do not involve $\forall$, then

$$S \vdash_C X \text{ if and only if } \sim\sim S \vdash_I \sim\sim X.$$

(Here $\sim\sim S$ is the result of prefixing $\sim\sim$ to each member of S.) Again, if we could produce an intuitionistic inner model in which (classically equivalent) forms of the set theory axioms not involving $\forall$ held, but $\sim$GCH also held (all prefixed with $\sim\sim$) we would have the classical independence of GCH. This can be done in a natural manner [3] and is, in form, very similar to the independence proof of Cohen [1] for GCH.

Yet another possibility is to use the modal logic S5. There are simple translations of classical logic into S5, for instance the following. Let X be a classical statement, define X° to be the result of replacing every subformula Y of X by $\Box Y$. Likewise, if S is a collection of statements, S° is the set of translates of members of S. Then

$$S \vdash_C X \text{ if and only if } S^\circ \vdash_{S5} X^\circ.$$

Now, an appropriate S5 inner model would establish classical independence of GCH. (I don't know if this can be done either.)

S4 can be used in a similar manner. Let us define X^* to be the result of replacing, in X, each subformula Y by $\Box\Diamond Y$, and define S^* analogously. Then it can be shown that

$$S \vdash_C X \text{ if and only if } S^* \vdash_{S4} X.$$

Here it doesn't matter whether S4 includes the Barcan formula or not [3]. Once again, a suitable S4 inner model would establish the classical independence of GCH. This can be done, and is also closely related to the proof of Cohen [1]. It has a special interest, because, as Gödel has observed [5], the S4 necessity operator has many essential properties of the notion "provable". Thus the translation used here amounts to replacing classical truth by provable consistency. It is an interesting notion and I wish I could present some satisfying reasons why it very likely ought to help in showing the classical independence of GCH apart from the fact that it actually does. Unfortunately, I can't.

Later on, in this paper, we will say more about these S4 inner models. But in order to place them in their proper setting we first discuss Gödel's construction of the inner model of constructible sets.

The intention of the axioms of set theory is to describe fully the intuitively conceived universe of sets. These are thought of as being all those sets one obtains by starting with the empty set and applying the basic operations of set theory "sufficiently often". The basic operations it suffices to consider are those of power set and union. But "sufficiently often" is most reasonably taken to mean α times, for any ordinal α, and so this becomes circular. Nevertheless, something may be made of this. Suppose V is some model for the axioms of set theory. We may define, for each ordinal $\alpha \in V$, a set R_α as follows.

$$R_0 = \emptyset;$$
$$R_{\alpha+1} = \text{power set of } R_\alpha;$$
$$R_\lambda = \cup_{\alpha<\lambda} R_\alpha \text{ (for } \lambda \text{ a limit ordinal).}$$

Then by our intuitive notions it is reasonble that V be exactly the collection of those sets x such that $x \in R_\alpha$ for some ordinal α. That is, that $V = \cup_\alpha R_\alpha$. In fact, this is often explicitly taken as an axiom of set theory; it is called the axiom of regularity.

Gödel's construction of an inner model is a modification of this. In a sense, it replaces the notion of power set of A by that of definable power set. (See [1] pp. 85—87.) We may make this precise as follows. Let A be some set. Let $P(x)$ be a formula of set theory with only x free. Call $P(x)$ *restricted to A* if

1) any constant of $P(x)$ denotes an element of A; and

2) the quantifiers of $P(x)$ are of the forms $(\forall y \in A)$ and $(\exists y \in \forall)$.

Then we may form a set B,

$$B = \{x \in A \mid P(x)\}.$$

Call B a *predicatively definable* subset of A. Let $\mathfrak{I}(A)$ be the collection of all predicatively definably subsets of A. Now, instead of the above R_α transfinite sequence, Gödel used the following M_α sequence:

$$M_0 = \varnothing;$$
$$M_{\alpha+1} = \mathfrak{I}(M_\alpha);$$
$$M_\lambda = \cup_{\alpha<\lambda} M_\alpha.$$

Then, L, the class of constructible sets, is the collection of those x such that $x \in M_\alpha$ for some ordinal α. It is not clear whether V and L should be the same according to our intuitive conception of set; nonetheless, L, whether or not it coincides with V, is a model in which all axioms of set theory, as well as GCH, hold.

This construction of L associates, in a natural way (the domain of) a classical logic model with each ordinal number. The model associated with $\alpha+1$ grows out of that for α in a reasonable manner. We want to generalize this procedure in a natural way to associate a (Kripke) S4 model with each ordinal. To simplify things we will only work with constant domain models, those in which the Barcan formula holds. Furthermore, the collection of possible worlds will be the same from ordinal to ordinal. We will use the notation $\langle G, R, \models_\alpha, S_\alpha\rangle$ to denote the S4 model associated with the ordinal α. Here G is a collection of possible worlds, R is a transitive, reflexive relation on G, S_α is a set of constants, and $\models_\alpha$ is a relation between possible worlds and statements with constants from S_α. We assume the usual S4 properties, namely: for each $\Gamma \in G$,

$$\Gamma \models_\alpha (X \wedge Y) \text{ if and only if } \Gamma \models_\alpha X \text{ and } \Gamma \models_\alpha Y;$$
$$\Gamma \models_\alpha \sim X \text{ if and only if not} \cdot \Gamma \models_\alpha X;$$
$$\Gamma \models_\alpha (\exists x)P(x) \text{ if and only if } \Gamma \models_\alpha P(c) \text{ for some } c \in s_\alpha;$$

$\Gamma \models_\alpha \Box X$ if and only if for each $\Box \in G$ such that $\Gamma R \Delta$, $\Delta \models_\alpha X$.

(We assume all formulas have $\in$ as the only predicate symbol.)

Now suppose, somehow, $\langle G, R, \models_\alpha, S_\alpha \rangle$ has been defined. We want to define $\langle G, R, \models_{\alpha+1}, S_{\alpha+1} \rangle$ in a natural way from it. Well, suppose $P(x)$ is a formula with all constants from S_α, and with only x free. We create a new constant, c_P, associated with $P(x)$. Let $S_{\alpha+1}$ consist of all such new constants, as well as all members of S_α. Now we define $\models_{\alpha+1}$ by specifying it for *atomic* statements of $S_{\alpha+1}$, which uniquely determines it for all such statements. Clearly what we want is something like: $\Gamma \models_{\alpha+1} (d \in c_P)$ if $\Gamma \models_\alpha P(d)$, where Γ is a possible world. The following states this precisely. For each $\Gamma \in G$,

1) if c, $d \in S_\alpha$, let $\Gamma \models_{\alpha+1} (d \in c)$ if $\Gamma \models_\alpha (d \in c)$;
2) if $d \in S_\alpha$ and $c_P \in S_{\alpha+1} - S_\alpha$, let $\Gamma \models_{\alpha+1} (d \in c_P)$ if $\Gamma \models_\alpha P(d)$;
3) if $d_P \in S_{\alpha+1} - S_\alpha$, let $\Gamma \models_{\alpha+1} (d_P \in c)$ provided there is some member $a \in s_\alpha$ such that $\Gamma \models_\alpha [(\forall x)(x \in a \equiv \equiv P(x))]^*$ and $\Gamma \models_{\alpha+1} (a \in c)$.

In clause 3) the '*' refers to the '$\Box\Diamond$'-translation of classical logic into S4 discussed earlier. The intent of this clause is to ensure that when we are done the axiom of extensionality will hold. Observe that the condition $\Gamma \models_{\alpha+1} (a \in c)$ is covered by clauses 1) and 2).)

Thus we have defined $\langle G, R, \models_{\alpha+1}, s_{\alpha+1} \rangle$ from $\langle G, R, \models_\alpha, s_\alpha \rangle$. Next, let λ be a limit ordinal. We simply let $s_\lambda = \cup_{\alpha<\lambda} s_\alpha$. We require that $\Gamma \models_\lambda (d \in c)$ if $\Gamma \models_\alpha (d \in c)$ for some $\alpha < \lambda$. Thus we specify $\langle G, R, \models_\lambda, s_\lambda \rangle$.

Finally we may define a limiting 'class' model. Let $s = \cup_{\alpha<\lambda} s_\alpha$. Let $\Gamma \models (d \in c)$ if $\Gamma \models_\alpha (d \in c)$ for some ordinal α. We have then specified our 'class' S4 model $\langle G, R, \models, s \rangle$, which is completely determined once an initial model, $\langle G, R, \models_0, s_0 \rangle$ is specified. This is a natural generalization of the Gödel M_α sequence. In fact, if $s_0 = \varnothing$ and G consists of a single possible world, the corresponding S4 sequence is only a notational variant of the M_α sequence.

We haven't yet said how all this is to begin, what we are to take for an initial model. We could, of course, take $s_0 = \emptyset$, but it is not necessary to be so restrictive. If we merely assume $\langle G, R, \models_0, s_0 \rangle$ is, itself, a set, and that '$\Box\Diamond$' translates of extensionality and regularity axioms hold in it, then all the techniques involved in showing L is a classical logic set theory model readily adapt to this sequence of S4 models, and demonstrate the following:

If $\langle G, R, \models_0, s_0 \rangle$ satisfies the conditions given above, and if X is any theorem of set theory, X^* is valid in $\langle G, R, \models, s \rangle$.

This is the S4 inner model generalization that we wanted.

We have considerable leeway in our choice of $\langle G, R, \models_0, s_0 \rangle$. All kinds of conditions may be built in at the start, so to speak. For instance, we may begin with an initial model with $\models_0$ having a great degree of symmetry. If this is properly done, in the corresponding class model the translate of the negation of the axiom of choice will be valid. Another choice of initial model provides a class model in which the translate of the axiom of choice is valid but the translate of $\sim$GCH is also. The essence of the construction here is to produce an initial model in which the constants which are to play the roles of $\aleph_0$, $\aleph_1$ and $\aleph_2$ in the limiting class model closely correspond to the sets which are $\aleph_0$, $\aleph_1$ and $\aleph_2$ in the model L, but the $\aleph_0$ constant has so many subsets that its power set will be at least $\aleph_2$ in the final class model. We do not give details here. In [2] specific such initial models are given for a corresponding intuitionistic logic generalization; the appropriate initial models for this S4 version are essentially the same. In either case the models are basically those of [1].

Thus, even though the classical logic inner models were demonstrably inadequate to establish the independence results referred to above, a reasonable generalization of the inner model method to include standard non-classical logics works very well. This is not the approach Cohen took, though it is basically the Scott-Solovay Boolean-valued model approach. Furthermore, all these approaches to the independence results of set theory are closely related.

Cohen [1] developed a notion called *forcing* and used it to construct classical (but not inner) models establishing independence. As has been observed by others [8, 10] his notion of forcing has a great deal of similarity with the notion of validity in Kripke intuitionistic logic models, and in [2] we carried out the independence proofs using exclusively these intuitionistic logic models. Cohen's method of constructing classical set theory models using the notion of forcing, when suitably applied, can be used to demonstrate the connections between intuitionistic logic, S4, and classical logic which we made use of above. In fact, the techniques of forcing and of non-classical logic independence proofs are the same. The primary difference between them is in the way one thinks about them, not in the mathematical technique.

The Scott and Solovay approach [see 11] may be looked at as another generalization of the inner model method. Instead of working with classical logic whose two truth-values form the simplest Boolean algebra, they work with logics whose truth-values form a more general Boolean algebra. Specifying a Boolean algebra then specifies a logic, not necessarily classical logic, but one closely related. For each such logic they define a transfinite sequence of models ending with a class model, again in a way that naturally generalizes the M_α sequence. No matter what the particular Boolean algebra chosen, in the resulting class model all theorems of set theory are valid. Then appropriate choices of Boolean algebras produce class models establishing the various independence results. The relationship between this approach and the S4 approach is more or less standard. There is a well-known algebraic model theory for S4, closure algebras, based on Boolean algebras. Standard connections between these closure algebras and Kripke S4 models provide the means for converting between Boolean-valued independence proofs and S4 independence proofs. Similar algebraic results provide the connection between Boolean-valued proofs and intuitionistic logic based proofs. Chapter 14 of [2] covers this in detail.

The S4 based independence proofs and the intuitionistic logic based independence proofs are quite simply related, via the standard embeddings of intuitionistic logic into S4. Thus all these approaches are equivalent in the sense that an independence proof based on one approach can be translated into an independence proof based on any of the other approaches discussed here. Nevertheless, each approach has its own reason for being; what is 'natural' using S4 models may not be using Boolean algebras, and vice versa. A multitude of possible approaches is a virtue. The reasonable question to conclude with is a necessarily vague one: Why have these particular inner model generalizations worked? What do they have in common? What is behind it all?

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