Propositional Logic Using Elementary Algebra

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1 Introduction: Propositional logic, developed using truth tables, is a familiar subject to many people; see [2] or [3]. It is not generally realized, however, that Boole's original treatment [1] was much closer to traditional algebra in motivaton and appearance than truth tables are. In fact, the subject can be developed while within the framework of elementary algebra, and doing so provides some interesting insights and exercises. In this paper we outline such a development. The development is not original with us; it is essentially part of the folklore of the subject, and deserves to be more widely known.

2 Basic idea: Propositional logic is *two-valued*: T (truth) and F (falsehood). To fit this into a more conventional framework, we will use two *numbers* for these values. The choice of which two is arbitrary, but 0 and 1 are by far the most obvious. Which is to be paired with T and which with F is also arbitrary; we have chosen to associate 0 with T and 1 with F. (The reader might find it instructive to parallel the development below using the opposite convention.) We use the following notation for the basic propositional connectives: $\wedge$ (and), $\vee$ (or), $\sim$ (not), $\supset$ (implies), $\equiv$ (if and only if). Now, the truth tables for these connectives, written with 0 in place of the customary T and 1 for F, are as follows:

X	Y	$X \wedge Y$	$X \vee Y$	$\sim X$	$X \supset Y$	$X \equiv Y$
0	0	0	0	1	0	0
0	1	1	0	1	1	1
1	0	1	0	0	0	1
1	1	1	1	0	0	0

Each connective is then an operation on the set $\{0,1\}$ and can be defined from the standard operations of arithmetic. There are many ways of doing this; the following are as simple as any.

$X \wedge Y$ is $X + Y - X \cdot Y$

$X \vee Y$ is $X \cdot Y$

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$\sim X$ is $1 - X$

$X \supset Y$ is $(1 - X) \cdot Y$

$X \equiv Y$ is $X + Y - 2X \cdot Y$

Using the above translation, each formula of logic translates into a conventional algebraic expression. For example,

$$\sim (A \wedge B) \supset (\sim A \vee \sim B) \tag{1}$$

translates, step by step, as follows:

$$\sim (A + B - AB) \supset [(1 - A) \vee (1 - B)]$$
$$[1 - (A + B - AB)] \supset [(1 - A) \cdot (1 - B)]$$
$$\{1 - [1 - (A + B - AB)]\} \cdot [(1 - A) \cdot (1 - B)] \tag{2}$$

Then, the question of whether (1) has the value T on every line of its truth table becomes the question of whether (2) is identically 0 as A and B range over the set $\{0,1\}$. This question, in practice, can be handled in two ways.

3 Algebraic approach: In the real number system the equation $X^2 = X$ is true if and only if X is 0 or 1. This characterizes the set $\{0,1\}$ over which our variables range. Thus, the problem of showing that (1) comes out T on every line of a truth table turns into the problem of showing that (2) is identically 0 *using the usual rules of elementary algebra together with the additional rule*: $X^2 = X$ *for variables*. (Appropriately, a *Boolean ring* is a ring meeting the additional condition $X^2 = X$.) For the particular formula in question, we can do this as follows:

$$\{1 - [1 - (A + B - AB)]\} \cdot [(1 - A)(1 - B)]$$
$$= \{1 - [1 - A - B + AB]\} \cdot [1 - A - B + AB]$$
$$= \{1 - 1 + A + B - AB\} \cdot [1 - A - B + AB]$$
$$\{A + B - AB\} \cdot [1 - A - B + AB]$$
$$A + B - AB - A^2 - AB + A^2B - AB - B^2 + AB^2 + A^2B + AB^2 - A^2B^2$$
$$\text{(since } X^2 = X \text{ for variables)} \tag{3}$$
$$= A + B - AB - A - AB + AB - AB - B + AB + AB + AB - AB$$
$$= 0$$

It follows that formula (1) is a tautology.

Such computations can often be shortened with a little ingenuity. We have that $X^2 = X$ for *variables*. It certainly does not follow that $X^2 = X$ for *every* algebraic expression. For example, even if A ranges over $\{0, 1\}$ $A + A$ ranges over $\{0,2\}$, so $(A + A)^2 = A + A$ is not always true. But, it is easy to see we do have $x^2 = X$ whenever X is an expression built from the variables using the algebraic counterparts of $\wedge$, $\vee$, $\sim$, $\supset$, and $\equiv$ given in ∯ 2. Then, the calculation above can be simplified as follows:
Truth value of formula (3)

$$= \{A + B - AB\} \cdot \{1 - [A + B - AB]\}$$

$$= [A + B - AB] - [A + B - AB]^2 \qquad \text{(multiplication)}$$

$$= [A + B - AB] - [A + B - AB]$$

$$= 0$$

Here we used $X^2 = X$ where X is $A + B - AB$, which is allowed since this is our algebraic translation for $A \wedge B$.

Starting with a formula which is *not* a tautology will lead to an expression which is not identically 0, and often some algebraic adroitness can actually produce counter-examples, that is, lines of a truth table falsifying the formula. As an example, consider

$$(A \vee B) \supset (A \wedge B) \tag{4}$$

which translates into

$$\begin{aligned} &(1 - AB)(A + B - AB) \\ &= A + B - AB - A^2B - AB^2 + A^2B^2 \\ &= A + B - AB - AB - AB + AB \\ &= A + B - 2AB \end{aligned} \tag{5}$$

and this certainly does not appear to be identically 0. Suppose we ask ourselves, when is it 0; that is, solve

$$A + B - 2AB = 0.$$

Well, since $X^2 = X$ for variables, we have

$$A^2 + B^2 - 2AB = 0$$

$$(A - B)^2 = 0$$

$$A - B = 0$$

$$A = B$$

Reading this argument backward, if $A \neq B$ then formula (5) is not 0. Thus, there are two counter-examples to (4): A is T, B is F; and A is F, B is T.

4 Arithmetic approach: We may show a formula of logic comes out T on every line of a truth table by numerically evaluating its translate for every assignment of 0 or 1 to the variables. If we always get 0, the formula is a tautology. This approach is quite routine, and the details may be carried out by a machine. The author will be happy to share his program for a Texas Instruments TI58 or TI59 programmable calculator with anyone requesting it.

REFERENCES:

1 Boole, George, *An Investigation of the Laws of Thought*, 1854, reprinted by Dover Publications, Inc., New York.

2 Kleene, S.C., *Mathematical Logic*, John Wiley and Sons, New York (1967).

3 Quine, W., *Mathematical Logic*, Revised Edition, Harper and Row, New York (1951).