# A Quantified Logic of Evidence

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**Abstract**

A propositional logic of explicit proofs, LP, was introduced in [2], completing a project begun long ago by Gödel, [13]. In fact, LP can be looked at in a more general way, as a logic of explicit evidence, and there have been several papers along these lines. A major result about LP is the Realization Theorem, that says any theorem of S4 can be converted into a theorem of LP by some replacement of necessitation symbols with explicit proof terms. Thus the necessitation operator of S4 can be seen as a kind of implicit existential quantifier: there exists a proof term (explicit evidence) such that. . . . In this paper, quantification over evidence is introduced into LP, and it is shown that the connection between S4 necessitation and the existential quantifier becomes an explicit one. The extension of LP with quantifiers is called QLP. A semantics and an axiom system for QLP are given, soundness and completeness are established, and several results are proved relating QLP to LP and to S4.

## 1 Introduction

Sometimes researchers work in different areas, for different purposes, and after a time it is discovered that there are strong common elements in their work. This is one of those cases. Thus I think it best to begin with a brief historical account, before describing the contribution of the present paper.

There is a long history of using modal logic to investigate provability in arithmetic, going back to Gödel. His published ideas in [12] about making $\Box$ correspond to provablity in Peano arithmetic eventually led to the modal logic GL, [5, 17, 6]. This is a remarkable logic with beautiful features. But Gödel also made a lesser-known proposal for the development of a logic of *explicit* proofs, as part of a general program to provide a foundation for arithmetic. This was not published in his lifetime, and was essentially unknown until it appeared in [13]. The idea of a logic of explicit proofs was independently found and successfully carried through to completion by Artemov, [1, 2]. The resulting logic is called LP, for 'logic of proofs.' Instead of a single modal operator, as in GL, it has infinitely many *proof terms*, with natural operations on them. There is an arithmetic completeness theorem for LP, relating it to formal arithmetic. There is also a natural relationship with the modal logic S4. One can think of LP as providing a fine-grain analysis of S4 validities.

There is also a long history of modal logics of knowledge, going back to [14], with [7] providing the (relatively) current state of things. The modal approach to modeling knowledge has had much success, but also some problems, most notably that of logical omniscience—in modal logics of knowledge we tend to know too much. One plausible way of minimizing such problems is to move from *known* to *known for a reason.* Then one might know the consequences of what one knows, but for more complex reasons, and this complexity of reasons provides machinery that might be used to circumscribe an agent's actual knowledge. Of course, mathematical proofs serve as explicit reasons for some of the things we know. The coincidence of different research areas that I mentioned at the start of this section, can now be stated. Use the ideas of LP as the basis of a logic of knowledge with explicit evidence. This was explored to some extent in [9].

The arithmetic semantics for LP is fundamental if one is interested in it strictly as a logic of proofs, but other semantics have also been created. There is a Kripke-style semantics in [10], which makes use of ideas from a simpler semantics in [15]. The situation is reminiscent of GL, which also has both an arithmetic interpretation and a Kripke-style semantics. A feature of the Kripke-style semantics for LP is that it makes connections with an S4 logic of knowledge seem tantalizingly close.

One important result about LP, briefly mentioned above, is the *realization* theorem, which connects the logic with S4. It says that, if one takes a theorem of S4, there is some way of replacing all occurrences of the $\Box$ operator with proof terms that will result in a theorem of LP (and conversely, though this is the easy direction). The original proof of this is in [2], and is proof-theoretic, making use of a cut-elimination result for a sequent calculus version of S4. Alternatively, there is a semantic proof in [10]. Indeed, semantical (non-constructive) proofs of cut elimination can also be given, [16, 10]. A full statement of the realization theorem is more nuanced, involving positive and negative occurrences of $\Box$, and other things, but the details can be skipped here.

In effect, the realization theorem involves quantification 'from the outside;' one thinks of the $\Box$ operator as a kind of quantifier, *there is a proof of.* It is reasonable to try and bring this quantification inside. And there has been work on adding quantifiers to LP, where quantification is over proofs (proof terms). In [18] such a project is carried out, and connections with arithmetic are investigated. Unfortunately, it is shown that this results in a non-axiomatizable logic. But one can come at the problem from another direction. Suppose we simply begin with axiomatically formulated LP, add the usual axiomatic machinery for quantification, and perhaps a little more as it seems appropriate. Then perhaps we can find a natural Kripke-style semantics to correspond to this, even if the connection with arithmetic is broken. We can think of what we are doing as part of the project to investigate the logic of explicit reasons, which presumably is broader than that of explicit proofs. Carrying out the approach just outlined constitutes the present paper.

Neither the axiomatization nor the semantics presented here is the first version. That can be found in [11]. The present version replaces the earlier one because this semantics is more natural, with fewer arbitrary features. I have moved the name QLP from the earlier presentation to this one, but there should be no confusion since I hope the earlier approach will fade away, leaving the designation of the name unambiguous. As noted, soundness and completeness results for QLP are shown. Using the semantic machinery, it is shown that S4 embeds in QLP in the expected way, translating $\Box$ by an explicit existential quantifier, "there exists a proof of." It is also shown that QLP is a conservative extension of LP.

Finally the semantics introduced for QLP, if scaled back to LP, still offers some new features. The semantics of both [15] and [10] are quite syntactic in nature. Proof terms themselves are part of the machinery. This has been treated more abstractly in the present paper. Each QLP model has a domain of reasons, with operations on them, and separately there is a mapping from proof terms to this domain. This treats terms more as they are treated in classical first-order logic. Of course among the various models there is a 'term model' in which the domain is actually a set of proof terms, and this is used to prove completeness. But the generalized notion of model allows for greater flexibility in proving results about the logic.

The other novel item is of rather lesser importance. In LP, as usually formulated, there are *proof constants* intended to represent proofs of simple facts, axioms. These proof constants have no structure. In particular, there is no connection between a proof constant for a formula $\varphi(x)$ and a proof constant for an instance, $\varphi(a)$, of it. Here proof constants are still allowed, but it is natural to extend the collection of proof constants to what I call *primitive proof terms*, which can have a bit more structure—they can contain variables. Then if $t(x)$ is a primitive proof term that represents a proof of $\varphi(x)$, we might arrange things so that $t(a)$ will represent a proof of $\varphi(a)$. Of course this additional machinery can be added to LP as well.

I would like to thank Evan Goris and other members of the Artemov seminar for helpful comments.

## 2 Syntax

Since QLP is a quantified version of LP, I'll begin with LP. Following [2], the collection of formulas of LP is built up from an infinite list of propositional letters and $\bot$, using $\supset$, and the following additional formation rule. If $X$ is a formula and $t$ is a proof term, then $t{:}X$ is a formula; it can be read "$t$ is a proof of $X$," or "$t$ is a reason for $X$," or "$t$ is evidence for $X$." To be neutral, it might be said that $t$ *verifies* $X$. Other propositional connectives are introduced as defined symbols, in the usual way. The collection of proof terms of LP is built up from an infinite list of proof variables (typically $x$, $y$, $x_1$, $x_2$, …) and proof constants (typically $c$, $d$, $c_1$, $c_2$, …), using the following formation rules. If $t$ and $u$ are proof terms, so are $t \cdot u$, $t + u$, and $!t$. Proof constants are meant to be justifications for obvious facts, such as logical axioms, which are not further analyzed. Proof variables are, well, variables. As for the operation symbols, the intuition is as follows. $t \cdot u$ is meant to be the result of joining together the two reasons $t$ and $u$; typically if $t$ justifies $X \supset Y$ and $u$ justifies $X$ then $t \cdot u$ justifies $Y$. $t + u$ is a kind of union or choice operation; it justifies what either $t$ or $u$ justifies. And ! is a verification operator; $!t$ verifies the correctness of an application of $t$. It is often called a *proof checker* operation. The axioms in the next section reflect all this directly.

Earlier I mentioned that I would be elaborating proof constants into more complex entities. As traditionally presented in the LP literature, constants are intended to be unanalyzed proofs of obvious facts, but apart from this they can be quite arbitrary. One can take a single constant for all axioms, or provide multiple constants for a single axiom, or anything inbetween. Now suppose $\varphi(x)$ is an LP axiom in which the variable $x$ occurs, and $\varphi(a)$ is a substitution instance of it, which will also be an axiom, since LP is formulated using axiom *schemes*. Both will have constants associated with them, but there need be no connection between the constants. However, once quantifiers have been introduced variables play a more fundamental role, and it is useful to have the possibility of reflecting substitution in axioms by substitution in their symbolic proofs. Consequently, I make the following modification to LP. Instead of just constants, the language contains function symbols of various arities (distinct from $\cdot$, $+$, and !, of course). These are called *primitive function symbols*. Then, if $f(x)$ serves to prove $\varphi(x)$, we may wish to arrange things so that $f(a)$ serves to justify $\varphi(a)$. Or we may not. The point is that the flexibility is there. The arity of a primitive function symbol may be 0, in which case it is a constant. Thus the current notion extends the usual use of proof constants.

**Definition 2.1** A *primitive proof term* is a proof term of the form $f(x_1, \ldots, x_n)$, where $f$ is a primitive function symbol and $x_1$, …, $x_n$ are proof variables. A *primitive term specification* is a mapping $\mathcal{F}$, assigning to each primitive proof term $p$ some set (possibly empty) of formulas. Think of $\mathcal{F}$ as mapping a primitive proof term to the set of formulas it potentially justifies. A formula $X$

*has a primitive proof term* with respect to $\mathcal{F}$ if $X \in \mathcal{F}(p)$ for some primitive proof term $p$. Likewise a primitive proof term $p$ is *for* a formula $X$ if $X \in \mathcal{F}(p)$.

I have made no assumptions about actual free variable occurrence. If $X \in \mathcal{F}(p)$, it is allowed that $p$ have free variables that do not occur in $X$, or that $X$ have free variables that do not occur in $p$. Doing so makes no difficulties for the theory, and provides some extra flexibility, though in practice one might want to impose some restrictions. As was noted earlier, primitive proof terms can involve primitive function symbols of arity 0, that is, they can be constants. So a constant specification, in the standard LP sense, is also a primitive term specification.

The formation rules for QLP extend those of LP, as follows. First, quantification is added: if $\varphi$ is a formula and $x$ is a proof variable, then $(\forall x)\varphi$ is a formula. The existential quantifier is defined in terms of the universal quantifier. Proof terms are built up from proof variables and primitive proof terms (rather than proof constants) using $\cdot$, ! and $+$, as before. In addition, QLP has one more operation on proof terms: if $t$ is a proof term and $x$ is a proof variable, then $(t \forall x)$ is a proof term. The intention is that $(t \forall x)$ should serve as a justification of $(\forall x)\varphi$ if $t$ serves as a *uniform* justification of each instance of $\varphi$. The usual conventions concerning free and bound occurrences are assumed, with occurrences of $x$ in $(t \forall x)$ considered bound. I will use the common convention of writing $\varphi(x)$ to indicate a formula with some (possibly no) free occurrences of the variable $x$, and $\varphi(a)$ to be the result of replacing all free occurrences of $x$ with occurrences of $a$, and similarly for proof terms—$t(x)$ becomes $t(a)$ on substitution of $a$ for $x$. The language of QLP thus defined will be referred to as $L^{\mathsf{QLP}}$. The sublanguage without the $\forall$ operator or quantifiers is the language of LP, with primitive proof terms replacing proof constants; it will be denoted $L^{\mathsf{LP}}$.

## 3 An Axiom System

The axiom system for QLP is based on that for LP. All of what follows are actually axiom schemes. I begin with the LP schemes, taken from [2].

1. A finite set of classical axiom schemas, sufficient for tautologies.
2. $t{:}(X \supset Y) \supset (s{:}X \supset (t \cdot s){:}Y)$
3. $t{:}X \supset X$
4. $t{:}X \supset\ !t{:}(t{:}X)$
5. $s{:}X \supset (s+t){:}X$ and $t{:}X \supset (s+t){:}X$

To these are added two standard universal quantification axioms.

6. $(\forall x)\varphi(x) \supset \varphi(t)$, for any proof term $t$ that is free for $x$ in $\varphi(x)$.
7. $(\forall x)(\psi \supset \varphi(x)) \supset (\psi \supset (\forall x)\varphi(x))$, where $x$ does not occur free in $\psi$.

And finally, what I call a *uniformity formula*. Assume $y$ does not occur free in $t$ or $\varphi$.

8. $(\exists y)y{:}(\forall x)t{:}\varphi \supset (t \forall x){:}(\forall x)\varphi$.

A word about axiom 8. To conclude we have a proof of $(\forall x)\varphi(x)$, it is not enough to have a proof of each instance, something we can express by $(\forall x)(\exists y)y{:}\varphi(x)$. The proof needed for any one instance may be quite different from that of other instances. In fact, concluding we have a proof of

$(\forall x)\varphi(x)$ whenever we have $(\forall x)(\exists y)y{:}\varphi(x)$ is a version of the $\omega$ rule. A better hypothesis would be not only that each instance of $\varphi(x)$ has a proof, but that we can produce a proof in a uniform way, $(\forall x)t{:}\varphi(x)$. That is, instead of simply having a proof, $(\exists y)$, we have a uniform recipe, $t$, which may have $x$ free in it, that provides us with proofs. But this is still not quite enough—$(\forall x)t{:}\varphi(x)$ should not only be true, but verifiable. We should have a proof of this, and that is what the antecedent of axiom 8 asserts. The consequent of the axiom is that we have a proof of $(\forall x)\varphi(x)$, which we can calculate from the uniform proof of instances of $\varphi(x)$, a calculation represented by $(t \forall x)$.

There are three rules of inference. The first two come from LP. For starters, there is a good old standby.

**Modus Ponens**

$$\frac{X, X \supset Y}{Y}$$

Following [10], I'll say a primitive term specification $\mathcal{F}$ is *axiomatically appropriate* if $\mathcal{F}$ provides primitive proof terms for exactly the axioms listed above. This amounts to a requirement that primitive proof terms serve to justify the obvious, which in this case are elementary logical truths. The next rule depends on the choice of $\mathcal{F}$, which is assumed to be axiomatically appropriate.

$\mathcal{F}$ **Necessitation Rule** if $X$ is an axiom and $X \in \mathcal{F}(p)$, one may conclude the following.

$$\overline{p{:}X}$$

Finally there is a variation on the standard universal generalization rule.

**Justified Universal Generalization Rule**

$$\frac{t{:}\varphi(x)}{(t \forall x){:}(\forall x)\varphi(x)}$$

I'll refer to the system with the axioms and rules give above as QLP *with primitive term specification* $\mathcal{F}$, where $\mathcal{F}$ is the primitive term specification used in the $\mathcal{F}$ necessitation rule. As usual, $Z$ is a theorem if it is the last line of a proof.

In the rest of this section various results are proved concerning QLP. In all of them it is assumed that $\mathcal{F}$ is an axiomatically appropriate primitive term specification, and $\mathcal{F}$ is used in applications of the Necessitation Rule.

In [2] an internalization result (Lifting Lemma) is shown for LP. That carries over rather easily to QLP. A stronger version will be shown below, as Proposition 3.4.

**Proposition 3.1** *If $X$ is a theorem of QLP, then for some proof term $p$ the formula $p{:}X$ is also a theorem.*

**Proof** The argument for this is by induction on proof length. Much of it is a direct carry-over of the corresponding proof for LP so rather than repeating things, I refer you to [2]. The only new item that concerns us is the justified universal generalization rule.

Suppose the rule has been used to conclude $(t \forall x){:}(\forall x)\varphi(x)$ from $t{:}\varphi(x)$, and the result is known for $t{:}\varphi(x)$, that is, there is a proof term $p$ such that $p{:}t{:}\varphi(x)$ is provable. Now proceed as follows. From $p{:}t{:}\varphi(x)$, conclude $t{:}\varphi(x)$, by axiom 3 (or just use the fact that we have a proof of $t{:}\varphi(x)$). By the rule, conclude $(t \forall x){:}(\forall x)\varphi(x)$, and from this, axiom 4, and modus ponens, conclude $!(t \forall x){:}(t \forall x){:}(\forall x)\varphi(x)$. ■

**Corollary 3.2** *The usual rule of universal generalization is a derived rule.*

**Proof** Suppose we have a proof of $\varphi(x)$. Using Proposition 3.1, for some $p$ there is a proof of $p:\varphi(x)$. Then by the justified universal generalization rule we have $(p \forall x):(\forall x)\varphi(x)$. Now we conclude $(\forall x)\varphi(x)$ using axiom 3 and modus ponens. ■

**Proposition 3.3** *Suppose $p(x_1,\ldots,x_n){:}\varphi(x_1,\ldots,x_n)$ has a QLP proof, and $t_1,\ldots,t_n$ are free for $x_1,\ldots,x_n$ respectively, in $\varphi(x_1,\ldots,x_n)$ and in $p(x_1,\ldots,x_n)$. Then $p(t_1,\ldots,t_n){:}\varphi(t_1,\ldots,t_n)$ also has a QLP proof (using the same primitive term specification).*

**Proof** Extend the proof of $p(x_1,\ldots,x_n){:}\varphi(x_1,\ldots,x_n)$ as follows.

1. $p(x_1,\ldots,x_n){:}\varphi(x_1,\ldots,x_n)$
2. $(\forall x_1)\ldots(\forall x_n)p(x_1,\ldots,x_n):\varphi(x_1,\ldots,x_n)$, using the derived universal generalization rule, Corollary 3.2, $n$ times
3. $p(t_1,\ldots,t_n){:}\varphi(t_1,\ldots,t_n)$, making use of axiom 6 and modus ponens $n$ times.

■

Here is the promised stronger version of Proposition 3.1. Something a bit more general could be shown, involving derivation from premises, but it will not be needed here.

**Proposition 3.4** *If $X$ is a theorem of QLP, then for some proof term $p$ the formula $p{:}X$ is also a theorem, where all free variables of $p$ are also free variables of $X$.*

**Proof** Suppose $X$ is a theorem of QLP. By Proposition 3.1 we have that $p{:}X$ is a theorem, for some proof term $p$. If $p$ has free variables that do not occur in $X$, substitute closed proof terms for them, a substitution which will not change $X$, and apply Proposition 3.3. I note that closed proof terms must exist. It may be that constants are among the primitive function symbols, in which case we can use them, but even if not, if $t$ has $x$ free, $(t \forall x)$ does not, so we can still produce closed proof terms. ■

Finally, an item that will be of use in proving completeness. It says the actual choice of free variables in formulas and terms is relatively unimportant.

**Definition 3.5** A primitive term specification $\mathcal{F}$ *admits variable replacement* if it meets the following condition: If $\varphi(x_1,\ldots,x_n) \in \mathcal{F}(p(x_1,\ldots,x_n))$, and $y_1,\ldots,y_n$ are variables that are free for $x_1,\ldots,x_n$ respectively in $\varphi$, then $\varphi(y_1,\ldots,y_n) \in \mathcal{F}(p(y_1,\ldots,y_n))$.

I'll say primitive term specification $\mathcal{F}'$ extends specification $\mathcal{F}$ provided, for each primitive term $p$, $\mathcal{F}(p) \subseteq \mathcal{F}'(p)$. It is straightforward to show that, for any primitive term specification $\mathcal{F}$ there is a smallest extension that admits variable replacement. I'll call this the *variable closure* of $\mathcal{F}$.

**Proposition 3.6** *Let $\mathcal{F}'$ be the variable closure of $\mathcal{F}$. Then $\mathcal{F}'$ is also axiomatically appropriate, and the same formulas are theorems of QLP using either $\mathcal{F}$ or $\mathcal{F}'$ as primitive term specification.*

**Proof** Since the axioms of QLP are instances of schemas, changing variables in an axiom yields another axiom, and hence $\mathcal{F}'$ is also axiomatically appropriate. That no more formulas are provable using $\mathcal{F}'$ than are provable using $\mathcal{F}$ is an immediate consequence of Proposition 3.3. ■

## 4 Semantics

The semantics for QLP is a version of the LP semantics of [10] with first-order machinery added, and the LP semantics of [10] in turn is a combination of Kripke modal semantics and an earlier LP semantics from [15]. The paper [9] provides additional background and motivation. It is probably a good idea to begin with an intuitive discussion of the semantics, before proceeding to the formalities.

Loosely speaking, what we are trying to capture is the classic notion of justified true belief. Following Hintikka, [14], a Kripke model is used to capture knowledge in an idealized sense, i.e. true belief. We take the Kripke model to have S4 properties, reflexivity and transitivity. One can consider the worlds of this model to be possible states of knowledge. If state $\Delta$ is accessible from state $\Gamma$, we might think of $\Delta$ as a state we could reach, starting at $\Gamma$, after further thought and investigation, by which point our knowledge may have increased. We know $X$ at a state $\Gamma$, at least in principle, if no additional investigation will contradict $X$, that is, if $X$ is true at all states accessible from $\Gamma$.

Next, associated with each state is a set of *reasons* or *evidences*. Since, intuitively, additional work can enlarge our collection of evidence, the association of reasons with states is permitted to be monotonic—constant domains are not required.

There is required to be an *evidence function* that associates a set of formulas with each reason, at each state. The idea is, if $X$ is associated with reason $r$ at state $\Gamma$ then $r$ could serve as possible evidence for $X$ at $\Gamma$. The evidence function is really an indicator of *relevance*—it does not say $r$ is a correct reason for $X$, merely that it is a reason to be considered. For example, if I assert that my cat is timid, personal observation would be a relevant piece of evidence for this assertion, whether or not my cat is actually timid. Anything said in the Stanford Encyclopedia of Philosophy would not be a relevant piece of evidence. On the other hand, if I assert that Kant said thus-and-so about the categorical imperative, the Stanford Encyclopedia would be relevant evidence and personal observation would not be, again whether or not what I assert is true. I should note that since variables may be present in formulas, values must be assigned to them. This will be done by a valuation function, so an evidence function must also take valuations into account. This adds a little complexity to the picture, but the fundamental ideas are as described above.

The essential connection between the notions just described is this. We will take $X$ to be known at state $\Gamma$ for reason $r$ if $X$ is known in the Hintikka sense, true at all accessible states, and $r$ is relevant evidence for $X$. That is, $X$ is believed, since it is the case at all states that my investigations may lead to, $X$ is true, since the actual state is among those I consider, and $X$ is justified, since it has $r$ as a reason—justified, true, belief.

As noted above, the assignment of sets of reasons to states is monotonic—additional investigation may affect truth but not relevance. For the same reason the evidence function is monotonic.

There are various operations on reasons. For instance, there is an operation that takes a reason for $X$ and a reason for $X \supset Y$ and produces a reason for $Y$. These operations all come from LP, and have been much discussed in the literature. What is different here is that the operations are applied in an abstract domain of reasons, and not to proof terms at the syntactic level.

Beyond the familiar LP operations there is machinery for quantification, where quantifiers in formulas range over reasons, and this certainly requires discussion. Let $t$ be a proof term. In a model we can think of $t$ as defining a mapping from reasons to reasons in the following way. Let $x$ be a variable, assign $x$ some reason as a value, then evaluate $t$ in the model. This is done in a way that is similar to the evaluation of terms in standard first-order models, so I'll suppress the details for now. Since $t$ may have several variables, we need to specify which one we are using for input; the notation $\langle \lambda x.t \rangle$ is ready-made for this purpose. Finally, the presence of other variables means the behavior of $\langle \lambda x.t \rangle$ can only be specified relative to an *environment*, a valuation $v$ which supplies values for variables other than $x$. What will actually be specified is the behavior of $\langle \lambda x.t \rangle^v$, where the valuation is made explicit. Now suppose $t$ provides us with reasons for instances of $\varphi(x)$. More precisely, suppose that at an arbitrary state $\Gamma$ of a model, for an arbitrary reason $r$ in the domain of $\Gamma$, applying the function $\langle \lambda x.t \rangle^v$ to $r$ yields a reason, also in the domain of $\Gamma$, for $\varphi(r)$. If this is the case not just at $\Gamma$ but at all states accessible from $\Gamma$, and for all reasons available at those states, we can consider that $t$ uniformly establishes $\varphi(x)$ no matter what $x$ may be. Then we should be able to calculate a reason for $(\forall x)\varphi(x)$ from $t$, or more precisely from the function $\langle \lambda x.t \rangle^v$. This calculated reason for the universal generalization is what $(t \forall x)$ is intended to represent. Of course making the details of all this precise will take some formal machinery, but I hope the intention is clear.

The remainder of this section is given over to a formal definition of the semantics.

A *frame* is a structure $\langle \mathcal{G}, \mathcal{R} \rangle$ where $\mathcal{R}$ is a binary relation on the non-empty set $\mathcal{G}$. Members of $\mathcal{G}$ are referred to as *states* or *worlds*, as usual. It will be assumed that $\mathcal{R}$ is reflexive and transitive, that is, the frame is one for S4.

A *domain function* on a frame $\langle \mathcal{G}, \mathcal{R} \rangle$ is a mapping $\mathcal{D}$ from members of $\mathcal{G}$ to non-empty sets, whose members are called *reasons*. Thus what counts as a reason depends on the state. It is assumed that domain functions are *monotonic*, that is, for $\Gamma, \Delta \in \mathcal{G}$, $\Gamma \mathcal{R} \Delta$ implies $\mathcal{D}(\Gamma) \subseteq \mathcal{D}(\Delta)$. This amounts to saying that reasons are not tenuous—they remain even as knowledge increases. Given a frame $\langle \mathcal{G}, \mathcal{R} \rangle$ and a domain function $\mathcal{D}$ on it, $\overline{\mathcal{D}}$ is the *frame domain*, and is defined to be $\cup_{\Gamma \in \mathcal{G}} \mathcal{D}(\Gamma)$.

Given a frame $\langle \mathcal{G}, \mathcal{R} \rangle$ and a domain function $\mathcal{D}$ on it, an *interpretation* $\mathcal{I}$ is a mapping meeting the following conditions. $\mathcal{I}$ assigns to each $n$-place primitive function symbol $f$ an $n$-place function $f^{\mathcal{I}} : \overline{\mathcal{D}}^n \to \overline{\mathcal{D}}$. $\mathcal{I}$ assigns to the one-place function symbol ! a mapping $!^{\mathcal{I}} : \overline{\mathcal{D}} \to \overline{\mathcal{D}}$. $\mathcal{I}$ assigns to the two-place function symbol $\cdot$ a binary operation $\cdot^{\mathcal{I}} : \overline{\mathcal{D}} \times \overline{\mathcal{D}} \to \overline{\mathcal{D}}$, and to $+$ a binary operation $+^{\mathcal{I}} : \overline{\mathcal{D}} \times \overline{\mathcal{D}} \to \overline{\mathcal{D}}$. Finally, and most complicated, $\mathcal{I}$ assigns to $\forall$ a mapping $\forall^{\mathcal{I}}$ from the function space of $\overline{\mathcal{D}}$ to $\overline{\mathcal{D}}$ itself; that is, $\forall^{\mathcal{I}} : (\overline{\mathcal{D}} \to \overline{\mathcal{D}}) \to \overline{\mathcal{D}}$.

Interpretations are required to meet certain closure conditions. Those that are easy to state are given here; one more comes later. It is required that, for each $\Gamma \in \mathcal{G}$, $\mathcal{D}(\Gamma)$ is closed under $f^{\mathcal{I}}$ for every primitive function symbol $f$. In particular, if $f$ is 0-place, a constant, then $f^{\mathcal{I}}$ must be in $\mathcal{D}(\Gamma)$ for all $\Gamma \in \mathcal{G}$. It is also required that each $\mathcal{D}(\Gamma)$ be closed under $\cdot^{\mathcal{I}}$, $!^{\mathcal{I}}$, and $+^{\mathcal{I}}$. A closure condition will be placed on $\forall^{\mathcal{I}}$ below, once more machinery has been introduced.

Suppose we have a structure $\langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$, where $\langle \mathcal{G}, \mathcal{R} \rangle$ is a frame, $\mathcal{D}$ is a domain function, and $\mathcal{I}$ is an interpretation. A *valuation* $v$ is a mapping from proof variables to members of $\overline{\mathcal{D}}$. It is not required that $v(x)$ be in $\mathcal{D}(\Gamma)$ for every $\Gamma \in \mathcal{G}$. As usual, a valuation $w$ is an $x$-variant of a valuation $v$ if $v$ and $w$ agree on all variables except possibly for $x$. I'll use the notation $v\binom{x}{r}$ for the $x$-variant of $v$ that maps $x$ to $r$.

**Definition 4.1** Let the structure $\langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$ be fixed. For an arbitrary valuation $v$, every proof term $t$ is mapped to a member of $\overline{\mathcal{D}}$, denoted $t^v$, by the following recursive rules.

1. $x^v = v(x)$ for $x$ a variable
2. $f(t_1, \ldots, t_n)^v = f^{\mathcal{I}}(t_1^v, \ldots, t_n^v)$ for $f$ a primitive function symbol
3. $(t \cdot u)^v = (t^v \cdot^{\mathcal{I}} u^v)$
4. $(t + u)^v = (t^v +^{\mathcal{I}} u^v)$
5. $(!t)^v = !^{\mathcal{I}}(t^v)$

6. (a) Suppose $t^w$ has been defined for all valuations $w$. For each variable $x$, define a mapping, $\langle \lambda x.t \rangle^v : \overline{\mathcal{D}} \to \overline{\mathcal{D}}$ as follows. For each $r \in \overline{\mathcal{D}}$, $\langle \lambda x.t \rangle^v(r)$ is $t^w$ where $w = v\binom{x}{r}$.
   (b) $(t \forall x)^v = \forall^{\mathcal{I}}(\langle \lambda x.t \rangle^v)$

We can now state the final closure condition for interpretations. Let $t$ be a proof term and $v$ be a valuation. Suppose that for every proof variable $y$ occurring in $t$, other than $x$, $v(y) \in \mathcal{D}(\Gamma)$. Then it is required that $(t \forall x)^v \in \mathcal{D}(\Gamma)$. Since $(t \forall x)^v$ is defined using $\forall^{\mathcal{I}}$ this is, indeed, a restriction on $\mathcal{I}$.

**Definition 4.2** A proof term is *meaningful* at state $\Gamma$ of $\mathcal{G}$ with respect to $v$ provided that for each variable $x$ that has a free occurrence in it, $v(x) \in \mathcal{D}(\Gamma)$. *Meaningful formulas* are characterized in a similar way.

It follows from the various closure conditions imposed on $\mathcal{I}$ that if a proof term or formula is meaningful at $\Gamma$ with respect to $v$, then for every proof term $t$ that occurs in it, $t^v \in \mathcal{D}(\Gamma)$.

An *evidence function* $\mathcal{E}$ is a mapping that assigns to each $\Gamma \in \mathcal{G}$, to each $r \in \overline{\mathcal{D}}$, and to each valuation $v$, a set $\mathcal{E}(\Gamma, r, v)$ of formulas of $L^{\mathsf{QLP}}$. Note that the range of $\mathcal{E}$ is syntactic: it consists of sets of formulas. Think of the members of $\mathcal{E}(\Gamma, r, v)$ as the formulas that $r$ provides possible evidence for, in state $\Gamma$, using $v$ to supply values for the free variables of the formulas. There is further discussion to be found in [9, 10] for a variant of the present notion having a more concrete version of reasons.

Special conditions are imposed on evidence functions, as follows. For all formulas $X$ and $Y$, for all states $\Gamma, \Delta \in \mathcal{G}$, for all reasons $r$ and $s$ in $\overline{\mathcal{D}}$, and for all valuations $v$:

1. If $r \notin \mathcal{D}(\Gamma)$ then $\mathcal{E}(\Gamma, r, v) = \emptyset$.
2. $\Gamma \mathcal{R} \Delta$ implies $\mathcal{E}(\Gamma, r, v) \subseteq \mathcal{E}(\Delta, r, v)$ (evidence is monotonic).
3. $(X \supset Y) \in \mathcal{E}(\Gamma, r, v)$ and $X \in \mathcal{E}(\Gamma, s, v)$ implies $Y \in \mathcal{E}(\Gamma, (r \cdot^{\mathcal{I}} s), v)$ (application).
4. If $X \in \mathcal{E}(\Gamma, r, v)$ and $t$ is any proof term such that $t^v = r$, then $t{:}X \in \mathcal{E}(\Gamma, !^{\mathcal{I}}(r), v)$ (proof checker).
5. $\mathcal{E}(\Gamma, r, v) \cup \mathcal{E}(\Gamma, s, v) \subseteq \mathcal{E}(\Gamma, (r +^{\mathcal{I}} s), v)$ (choice).
6. Let $t$ be a proof term of $L^{\mathsf{QLP}}$, and suppose the following: for every $\Delta \in \mathcal{G}$ with $\Gamma \mathcal{R} \Delta$, and for every $r \in \mathcal{D}(\Delta)$, $X \in \mathcal{E}(\Delta, \langle \lambda x.t \rangle^v(r), v\binom{x}{r})$. Then $(\forall x)X \in \mathcal{E}(\Gamma, (t \forall x)^v, v)$.
7. If $v$ and $w$ agree on the free variables of $X$, then $X \in \mathcal{E}(\Gamma, r, v)$ iff $X \in \mathcal{E}(\Gamma, r, w)$

Condition 1 says that unavailable evidence explains nothing. Conditions 2–5 are modified from [10] and derive ultimately from [15]. The important new item is condition 6, which has a complicated appearance but is straightforward underneath. It says that $(t \forall x)^v$ serves as a possible reason for $(\forall x)X$ if $t$ provides us with (uniform) reasons for each instance of $X$. But, what instances $X$ has depends on what reasons are available to serve as values of the variable $x$, and these can vary from state to state. Consequently we must not only check $\Gamma$ itself, but every state accessible from $\Gamma$. If $\Delta$ is such a state, we need to have that $t$ provides us with a possible reason for $X$, with $x$ taking on any value from $\mathcal{D}(\Delta)$, say $r$. Saying that $x$ has $r$ as its value means we work with the valuation $v\binom{x}{r}$. And the reason that $t$ supplies us with is given by applying the function $\langle \lambda x.t \rangle^v$ to $r$. All this translates into the requirement that $X \in \mathcal{E}(\Delta, \langle \lambda x.t \rangle^v(r), v\binom{x}{r})$.

**Definition 4.3** A *weak QLP model* is a structure $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E}, \mathcal{V} \rangle$ where: $\langle \mathcal{G}, \mathcal{R} \rangle$ is a frame; $\mathcal{D}$ is a domain function, $\mathcal{I}$ is an interpretation, $\mathcal{E}$ is an evidence function, all meeting the conditions given above, and $\mathcal{V}$ is a mapping of propositional letters to sets of states.

Truth of formulas at worlds of weak model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E}, \mathcal{V} \rangle$, with respect to a valuation $v$, is evaluated in a way that extends that of [10]. The idea behind clause 5 below is that $X$ is potentially known with $t$ as justification at state $\Gamma$ if $X$ is known at $\Gamma$ (true at all accessible states), and $t$ serves as evidence for $X$ at $\Gamma$.

1. $\mathcal{M}, \Gamma \Vdash_v P \iff \Gamma \in \mathcal{V}(P)$ for $P$ a propositional letter;
2. $\mathcal{M}, \Gamma \not\Vdash_v \bot$;
3. $\mathcal{M}, \Gamma \Vdash_v X \supset Y \iff \mathcal{M}, \Gamma \not\Vdash_v X$ or $\mathcal{M}, \Gamma \Vdash_v Y$;
4. $\mathcal{M}, \Gamma \Vdash_v (\forall x)\varphi \iff \mathcal{M}, \Gamma \Vdash_w \varphi$ for every $w$ where $w = v\binom{x}{r}$ and $r \in \mathcal{D}(\Gamma)$.
5. $\mathcal{M}, \Gamma \Vdash_v (t{:}X) \iff X \in \mathcal{E}(\Gamma, t^v, v)$ and $\mathcal{M}, \Delta \Vdash_v X$ for every $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{R} \Delta$.

Note that if $t^v \notin \mathcal{D}(\Gamma)$ then $\mathcal{M}, \Gamma \not\Vdash_v t{:}X$ by condition 5, because $\mathcal{E}(\Gamma, t^v, v) = \emptyset$, using condition 1 on evidence functions. I'll say $X$ is *true at state* $\Gamma$ with respect to $v$ if $\mathcal{M}, \Gamma \Vdash_v X$, and otherwise $X$ is *false at* $\Gamma$. Finally, $X$ is *valid* in the structure $\mathcal{M}$ if, for every valuation $v$, $X$ is true at all states $\Gamma$ of $\mathcal{M}$ at which $X$ is meaningful with respect to $v$.

A weak model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E}, \mathcal{V} \rangle$ *meets primitive term specification* $\mathcal{F}$ provided that, for each primitive proof term $p$, each valuation $v$, and each $\Gamma \in \mathcal{G}$ it is the case that $\mathcal{F}(p) \subseteq \mathcal{E}(\Gamma, p^v, v)$.

A weak model meets the *fully explanatory* condition provided that, whenever $X$ is meaningful at state $\Gamma$ with respect to a valuation $v$, and $\mathcal{M}, \Delta \Vdash_v X$ for every $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{R} \Delta$, then $X \in \mathcal{E}(\Gamma, r, v)$, for some $r \in \mathcal{D}(\Gamma)$. Informally, the fully explanatory condition says that if $X$ is known at state $\Gamma$, in the Hintikka sense, then $X$ has a reason. A weak model that meets the fully explanatory condition is called *strong*.

**Proposition 4.4** *Suppose $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E}, \mathcal{V} \rangle$ is a weak model. If the valuations $v$ and $w$ agree on the free variables of $X$ then $\mathcal{M}, \Gamma \Vdash_v X$ iff $\mathcal{M}, \Gamma \Vdash_w X$, for every $\Gamma \in \mathcal{G}$.*

**Proof** The proof is via a standard induction on the complexity of $X$. Condition 7 on evidence functions is needed for the case where $X$ is $t{:}Y$. ■

In Sections 5 and 6 the soundness and completeness result stated below is proved. In subsequent sections the semantics will be used to establish various properties of QLP.

**Theorem 4.5** *Let $\mathcal{F}$ be a primitive term specification that is axiomatically appropriate (definition in Section 3). A formula $X$ of $L^{\mathsf{QLP}}$ has an axiomatic proof using specification $\mathcal{F}$ if and only if $X$ is valid in all weak QLP models that meet specification $\mathcal{F}$ if and only if $X$ is valid in all strong QLP models that meet specification $\mathcal{F}$.*

## 5 Soundness

It will be proved that axiomatic QLP is sound with respect to the weak model semantics, and hence also with respect to the strong model semantics. More specifically, what will be proved is this.

**Theorem 5.1** *Let $\mathcal{F}$ be an axiomatically appropriate primitive term specification. If $X$ has a* QLP *proof using the $\mathcal{F}$ necessitation rule then $X$ is valid in all weak* QLP *models that meet specification $\mathcal{F}$.*

This theorem is shown by induction on the length of the proof of $X$. Each axiom is valid in all weak QLP models. For tautologies this is obvious. The validity of universal instantiation, axiom 6, makes use of the facts that quantification at a state is over the domain of that state, that we only consider instantiation terms $t$ that are meaningful, and that state domains meet various closure conditions. The other quantificational axiom, 7, is straightforward. For axioms 2–5 the arguments are essentially the same as in [10]. I'll do one case in detail as an example: axiom 4. Suppose $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E}, \mathcal{V} \rangle$ is a weak QLP model, and $t{:}X \supset\, !t{:}t{:}X$ is meaningful at $\Gamma \in \mathcal{G}$ with respect to valuation $v$. I'll show $\mathcal{M}, \Gamma \Vdash_v t{:}X \supset\, !t{:}t{:}X$. Assume $\mathcal{M}, \Gamma \Vdash_v t{:}X$. Then $X \in \mathcal{E}(\Gamma, t^v, v)$ and $\mathcal{M}, \Delta \Vdash_v X$ for all $\Delta$ with $\Gamma \mathcal{R} \Delta$. Let $\Omega \in \mathcal{G}$ be an arbitrary state with $\Gamma \mathcal{R} \Omega$. Since $\mathcal{R}$ is transitive, $\mathcal{M}, \Delta \Vdash_v X$ for all $\Delta$ with $\Omega \mathcal{R} \Delta$. By the monotonicity of evidence functions, condition 2, $X \in \mathcal{E}(\Omega, t^v, v)$. It follows that $\mathcal{M}, \Omega \Vdash_v t{:}X$. Also, since $X \in \mathcal{E}(\Gamma, t^v, v)$, by condition 4 on evidence functions $t{:}X \in \mathcal{E}(\Gamma, !^{\mathcal{I}}(t^v), v) = \mathcal{E}(\Gamma, (!t)^v, v)$. Since $\Omega$ was arbitrary, $\mathcal{M}, \Gamma \Vdash_v !t{:}t{:}X$.

Axiom 8 needs some discussion. Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E}, \mathcal{V} \rangle$ be a weak QLP model with $\Gamma \in \mathcal{G}$, and suppose $\mathcal{M}, \Gamma \Vdash_v (\exists y) y{:}(\forall x) t{:}\varphi$, where $y$ does not occur free in $t$ or in $\varphi$. I'll show $\mathcal{M}, \Gamma \Vdash_v (t \forall x){:}(\forall x)\varphi$. By the hypothesis, for some $d \in \mathcal{D}(\Gamma)$, $\mathcal{M}, \Gamma \Vdash_w y{:}(\forall x) t{:}\varphi$, where $w = v\binom{y}{d}$. Let $\Delta$ be an arbitrary member of $\mathcal{G}$ with $\Gamma \mathcal{R} \Delta$. By condition 5 for truth in models, $\mathcal{M}, \Delta \Vdash_w (\forall x) t{:}\varphi$ and, since $y$ does not occur free in $t$ or $\varphi$, we have $\mathcal{M}, \Delta \Vdash_v (\forall x) t{:}\varphi$, by Proposition 4.4. Then for every $r \in \mathcal{D}(\Delta)$, $\mathcal{M}, \Delta \Vdash_{v'} t{:}\varphi$ where $v' = v\binom{x}{r}$. Since $\mathcal{R}$ is reflexive, $\mathcal{M}, \Delta \Vdash_{v'} \varphi$ and, since $r$ was arbitrary, $\mathcal{M}, \Delta \Vdash_v (\forall x)\varphi$. Again, since $\mathcal{M}, \Delta \Vdash_{v'} t{:}\varphi$, we must have $\varphi \in \mathcal{E}(\Delta, t^{v'}, v')$, or equivalently, $\varphi \in \mathcal{E}(\Delta, \langle \lambda x.t \rangle^v(r), v\binom{x}{r})$. Since $\Delta$ was arbitrary, by condition 6 on evidence functions, $(\forall x)\varphi \in \mathcal{E}(\Gamma, (t \forall x)^v, v)$. It follows that $\mathcal{M}, \Gamma \Vdash_v (t \forall x){:}(\forall x)\varphi$.

This leaves the three rules of inference. Call a rule *sound* provided that, whenever the premises are valid in all weak QLP models meeting primitive term specification $\mathcal{F}$, so is the conclusion. We must show soundness of each rule. For the following, let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E}, \mathcal{V} \rangle$ be a weak QLP model meeting primitive term specification $\mathcal{F}$.

For the $\mathcal{F}$ Necessitation Rule, let us suppose $X$ is an axiom and $X \in \mathcal{F}(p)$. Suppose that $p{:}X$ is meaningful at $\Gamma \in \mathcal{G}$ with respect to valuation $v$; I'll show that $\mathcal{M}, \Gamma \Vdash_v p{:}X$. Let $\Delta \in \mathcal{G}$ be an arbitrary state with $\Gamma \mathcal{R} \Delta$. A simple argument using the monotonicity of the domain function, $\mathcal{D}$, shows that $X$ must be meaningful at $\Delta$ with respect to $v$, and hence $\mathcal{M}, \Delta \Vdash_v X$ since axioms are weakly valid. Since $\mathcal{M}$ meets primitive term specification $\mathcal{F}$, $X \in \mathcal{F}(p) \subseteq \mathcal{E}(\Gamma, p^v, v)$. It follows that $\mathcal{M}, \Gamma \Vdash_v p{:}X$.

Modus Ponens has one twist that requires some attention. Suppose both $X$ and $X \supset Y$ are valid in $\mathcal{M}$; I'll show $Y$ is also valid. Let $\Gamma \in \mathcal{G}$ be a state at which $Y$ is meaningful with respect to $v$, but $\mathcal{M}, \Gamma \nVdash_v Y$; I'll derive a contradiction. The problem is that $X$ may have variables that do not occur in $Y$, and so $X$ and $X \supset Y$ might not be meaningful at $\Gamma$ with respect to $v$. Say $x_1$, …, $x_n$ are all the variables in $X$ that are not in $Y$. Choose $a_1$, …, $a_n \in \mathcal{D}(\Gamma)$, and let $w$ be the valuation that agrees with $v$ on all variables, except that $w(x_i) = a_i$, for $i = 1, \ldots, n$. Since $x_1$, …, $x_n$ do not occur in $Y$, we have $\mathcal{M}, \Gamma \nVdash_w Y$, and of course $Y$ is also meaningful at $\Gamma$ with respect to $w$. But also, $X$ and $X \supset Y$ are meaningful at $\Gamma$ with respect to $w$, so by our validity assumption, $\mathcal{M}, \Gamma \Vdash_w X$ and $\mathcal{M}, \Gamma \Vdash_w X \supset Y$, and a contradiction is immediate.

Finally we have the Justified Universal Generalization Rule. Suppose $t{:}\varphi$ is valid in $\mathcal{M}$. Let $\Gamma \in \mathcal{G}$ and let $v$ be a valuation such that $(t \forall x){:}(\forall x)\varphi$ is meaningful at $\Gamma$ with respect to $v$; I'll show

$\mathcal{M}, \Gamma \Vdash_v (t \forall x){:}(\forall x)\varphi$.

Let $\Delta$ be an arbitrary member of $\mathcal{G}$ with $\Gamma\mathcal{R}\Delta$. Since $(t\forall x){:}(\forall x)\varphi$ is meaningful at $\Gamma$, $v(y) \in \mathcal{D}(\Gamma)$ for all free variables $y$ in $t$ or $\varphi$, except possibly for $x$, and by the monotonicity of the domain function, $v(y) \in \mathcal{D}(\Delta)$ as well. Then if $r$ is an arbitrary member of $r \in \mathcal{D}(\Delta)$, $t{:}\varphi$ is meaningful at $\Delta$ with respect to $v\binom{x}{r}$. Then since $t{:}\varphi$ is valid, $\mathcal{M}, \Delta \Vdash_{v\binom{x}{r}} t{:}\varphi$. It follows from condition 5 for truth in models that we must have $\varphi \in \mathcal{E}(\Delta, t^{v\binom{x}{r}}, v\binom{x}{r}) = \mathcal{E}(\Delta, \langle \lambda x.t\rangle^v(r), v\binom{x}{r})$. Since $\Delta$ and $r$ were arbitrary, condition 6 on evidence functions gives us that $(\forall x)\varphi \in \mathcal{E}(\Gamma, (t \forall x)^v, v)$. In addition, for arbitrary $\Delta$ accessible from $\Gamma$ and for arbitrary $r \in \mathcal{D}(\Delta)$, since we have $\mathcal{M}, \Delta \Vdash_{v\binom{x}{r}} t{:}\varphi$ we must have $\mathcal{M}, \Delta \Vdash_{v\binom{x}{r}} \varphi$ (using reflexivity of $\mathcal{R}$) and hence $\mathcal{M}, \Delta \Vdash_v (\forall x)\varphi$. We have now established that $\mathcal{M}, \Gamma \Vdash_v (t \forall x){:}(\forall x)\varphi$.

## 6 Completeness

It will be proved that axiomatic QLP is complete with respect to the strong model semantics, and hence also with respect to the weak model semantics. Here is a proper statement of the result.

**Theorem 6.1** *Let $\mathcal{F}$ be a primitive term specification that is axiomatically appropriate. If a formula $X$ of $L^{\mathsf{QLP}}$ is valid in all strong QLP models that meet specification $\mathcal{F}$, then $X$ has an axiomatic proof using $\mathcal{F}$.*

The rest of the section is devoted to a proof of this Theorem. Let $\mathcal{F}$ be an axiomatically appropriate primitive term specification, fixed for the rest of the section. Without loss of generality, we assume $\mathcal{F}$ admits variable replacement, appealing to Proposition 3.6. As usual, completeness will be shown in contrapositive form: a formula not having a QLP proof using specification $\mathcal{F}$ is invalidated in some model meeting specification $\mathcal{F}$.

Let $p_1$, $p_2$, ..., be an infinite list of variables that are new to $L^{\mathsf{QLP}}$. I'll call these *parameters*. Let $L^*$ be the extension of $L^{\mathsf{QLP}}$ in which parameters are allowed to appear in formulas but: 1. parameters are never quantified, and 2. parameters do not appear as $x$ in the proof term $(t \forall x)$. The expanded language, and sublanguages of it, will be used to construct a model.

The interest is in theorems, and proofs, in the extended language $L^*$, but the primitive term specification $\mathcal{F}$ is only axiomatically appropriate for the original language $L^{\mathsf{QLP}}$. So we extend $\mathcal{F}$ to an axiomatically appropriate primitive term specification $\mathcal{F}^*$ for $L^*$, as follows. Suppose $\varphi(p_1, \ldots, p_k)$ is an instance of one of the axiom schemes for QLP, where this is a formula of $L^*$ and $p_1$, ..., $p_k$ are all the parameters it contains. Let $x_1$, ..., $x_k$ be $L^{\mathsf{QLP}}$ variables that do not occur in $\varphi$, and consider the formula $\varphi(x_1, \ldots, x_k)$, which is a formula of the original language $L^{\mathsf{QLP}}$, and again an instance of an axiom scheme of QLP. Then $\varphi(x_1, \ldots, x_k)$ has one or more primitive proof terms, let $f(x_1, \ldots, x_k)$ be any of them, and set $\varphi(p_1, \ldots, p_k) \in \mathcal{F}^*(f(p_1, \ldots, p_k))$. The actual choice of variables $x_1$, ..., $x_k$ does not matter since we are assuming $\mathcal{F}$ admits variable replacement. The primitive term specification $\mathcal{F}^*$ extends $\mathcal{F}$, because if a formula contains no parameters, the process just described will assign the same primitive proof terms to the formula that $\mathcal{F}$ itself provides. Obviously $\mathcal{F}^*$ is axiomatically appropriate, but now with respect to $L^*$. When giving proofs of formulas from $L^*$, it is $\mathcal{F}^*$ that will be used and not $\mathcal{F}$ in applications of the necessitation rule.

Let us say a formula or a term is *pseudo-closed* if the only free variables it contains are parameters. If $\varphi(p_1, \ldots, p_k)$ is pseudo-closed and is an instance of one of the QLP axioms, $\mathcal{F}^*$ provides primitive proof terms for it, but from the definition of $\mathcal{F}^*$, it could happen that some of these are not pseudo-closed. According to the way $\mathcal{F}^*$ was defined, we consider $\varphi(x_1, \ldots, x_n)$, where the $x_i$ are fresh $L^{\mathsf{QLP}}$ variables, take any primitive proof term for this, given by $\mathcal{F}$, and replace the $x_i$ in it with $p_i$. But that primitive proof term could be $f(x_1, \ldots, x_k, y_1, \ldots, y_n)$, where the $y_i$ are additional proof variables of $L^{\mathsf{QLP}}$ that do not occur free in $\varphi(x_1, \ldots, x_k)$. Primitive term assignments are allowed to be quite general, and this is not ruled out. But then, since $\mathcal{F}$ admits variable replacement, $f(x_1, \ldots, x_k, x_1, \ldots, x_1)$ will also be a primitive proof term for $\varphi(x_1, \ldots, x_k)$ using $\mathcal{F}$, and if we use that, we get a pseudo-closed primitive proof term that $\mathcal{F}^*$ also provides for $\varphi(p_1, \ldots, p_k)$. So we can assume that $\mathcal{F}^*$ always provides pseudo-closed primitive proof terms for pseudo-closed formulas, indeed, terms whose parameters all occur in the formulas.

The following can be proved. I'll leave the argument to you—it is here that the formulation of $\mathcal{F}^*$ comes into play, as well as the assumption that $\mathcal{F}$ admits variable replacement.

**Lemma 6.2** *Suppose $\varphi(p)$ is a formula of $L^*$, where $p$ is a parameter (not necessarily the only one appearing in the formula). And suppose $\varphi(p)$ has a proof in the extended language, using primitive term specification $\mathcal{F}^*$. Let $x$ be a variable of $L^{\mathsf{QLP}}$ that does not occur in the proof. Replacing occurrences of $p$ throughout the proof by occurrences of $x$ produces another correct proof with respect to $\mathcal{F}^*$, of $\varphi(x)$.*

If $P$ is a set of parameters, by $L^*(P)$ is meant the sublanguage of $L^*$ with parameters restricted to the set $P$. Let $S$ be a set of pseudo-closed formulas from $L^*(P)$. $S$ is $L^*(P)$ *inconsistent* if there is a finite set $\{X_1, \ldots, X_n\} \subseteq S$ such that $X_1 \supset (X_2 \supset \cdots (X_n \supset \bot) \cdots)$ has a proof in $\mathsf{QLP}$ using the expanded language $L^*(P)$. $S$ is $L^*(P)$ *consistent* if it is not $L^*(P)$ inconsistent. $S$ is *maximally consistent* with respect to $L^*(P)$ if it is $L^*(P)$ consistent, and no proper superset of $S$ consisting of pseudo-closed formulas from $L^*(P)$ is $L^*(P)$ consistent. Finally, $S$ is $\exists$-*complete* with respect to $L^*(P)$ if, for every formula $\neg(\forall x)\varphi(x) \in S$ there is some pseudo-closed proof term $t$ in $L^*(P)$ such that $\neg\varphi(t) \in S$.

A few remarks are in order concerning this notion of consistency. Suppose $P$ and $P'$ are sets of parameters with $P \subseteq P'$, and $S$ is a set of pseudo-closed formulas from $L^*(P)$. Then $S$ is $L^*(P)$ consistent if and only if $S$ is $L^*(P')$ consistent. One direction is trivial. If $S$ is not $L^*(P)$ consistent, a finite subset of $S$ entails $\bot$ using the machinery of $L^*(P)$, and all that machinery is also available in $L^*(P')$. For the other direction, suppose $S$ is $L^*(P')$ inconsistent, and so there is a proof of $X_1 \supset (X_2 \supset \cdots (X_n \supset \bot) \cdots)$ in $L^*(P')$, where $\{X_1, \ldots, X_n\} \subseteq S$. Say $p_1, \ldots, p_k$ are all the parameters that occur in this proof that are in $P'$ but not in $P$. Let $y_1, \ldots, y_k$ be variables that do not occur in the proof, and replace all occurrences of $p_i$ by $y_i$ for each $i$. The result will still satisfy the conditions for being a proof, using Lemma 6.2, but it is a now a proof whose formulas are in $L^*(P)$. It follows that $S$ is $L^*(P)$ inconsistent as well.

**Proposition 6.3** *Let $S$ be a subset of $L^*(P)$ consisting of pseudo-closed formulas, that is $L^*(P)$ consistent, and suppose $\neg(\forall x)\varphi(x) \in S$. Also, let $p$ be a parameter not in $P$, and let $P' = P \cup \{p\}$. Then $S \cup \{\neg\varphi(p)\}$ is $L^*(P')$ consistent.*

**Proof** This is a familiar argument. Suppose, under the hypotheses of the Proposition, that $S \cup \{\neg\varphi(p)\}$ is not $L^*(P')$ consistent. Then for some finite subset $\{X_1, \ldots, X_n\}$ of $S$, $(X_1 \wedge \ldots \wedge X_n) \supset \varphi(p)$ is provable in $L^*(P')$. Let $z$ be a variable of $L^{\mathsf{QLP}}$ that does not appear in this proof. By Lemma 6.2 it follows that $(X_1 \wedge \ldots \wedge X_n) \supset \varphi(z)$ is also provable, within $L^*(P)$. Then using the derived universal generalization rule (Corollary 3.2) and axiom 7, $(X_1 \wedge \ldots \wedge X_n) \supset (\forall z)\varphi(z)$ is provable within $L^*(P)$, and hence so is $(X_1 \wedge \ldots \wedge X_n) \supset (\forall x)\varphi(x)$. But this is a contradiction since $X_1, \ldots, X_n \in S$, $\neg(\forall x)\varphi(x) \in S$, and $S$ is $L^*(P)$ consistent. ■

Now a standard Henkin construction gives us the following. I omit the details.

**Proposition 6.4** *Suppose $S$ is a consistent subset of $L^*(P)$ consisting of pseudo-closed formulas, where $P$ omits infinitely many parameters. Then there is an extension $P'$ of $P$ that also omits infinitely many parameters, and an extension $S'$ of $S$ in the language $L^*(P')$, consisting of pseudo-closed formulas, such that $S'$ is both maximally $L^*(P')$ consistent, and $\exists$-complete with respect to $L^*(P')$.*

If a set $S$ of pseudo-closed formulas is maximally consistent with respect to some some language $L^*(P)$, the set $P$ of parameters is uniquely determined by $S$. No parameter outside $P$ can appear in $S$, and every parameter in $P$ must appear since, for instance, $p{:}A \vee \neg p{:}A$ must appear for each $p \in P$ by maximality (here $A$ is any propositional letter).

I now describe the *canonical model*, a rather lengthy description. Call a set $S$ of pseudo-closed formulas of $L^*$ *$P$-world-like* if $S$ is maximally consistent and $\exists$-complete with respect to $L^*(P)$, where $P$ is a set of parameters that omits infinitely parameters. And call $S$ simply *world-like* if it is $P$-world-like for some $P$. As noted above, given a world-like set, the corresponding set of parameters is uniquely determined. Let $\mathcal{G}$ be the collection of all world-like sets of formulas.

For a set $\Gamma$ of formulas, let $\Gamma^\sharp = \{Z \mid t{:}Z \in \Gamma \text{ for some } t\}$. Say $\Gamma, \Delta \in \mathcal{G}$; we take $\Gamma\mathcal{R}\Delta$ provided $\Gamma^\sharp \subseteq \Delta$. We now have a frame, $\langle \mathcal{G}, \mathcal{R}\rangle$, and it must be verified that it is reflexive and transitive. For reflexivity it must be shown that $\Gamma^\sharp \subseteq \Gamma$. Well, suppose $\Gamma$ is $P$-world-like, and $Z \in \Gamma^\sharp$. Then $t{:}Z \in \Gamma$, for some $t$ in the language $L^*(P)$. The axiom $t{:}Z \supset Z$ (an instance of axiom scheme 3) is in the language $L^*(P)$, so it must be in $\Gamma$. Since maximal consistent sets are closed under modus ponens, then $Z \in \Gamma$. Thus reflexivity has been shown. Showing transitivity comes down to showing that $\Gamma^\sharp \subseteq \Gamma^{\sharp\sharp}$, and this is established similarly, using axiom schema 4.

**Lemma 6.5** *Let $\mathcal{G}$ and $\mathcal{R}$ be as defined above. Let $\Gamma \in \mathcal{G}$, where $\Gamma$ is $P$-world-like, and suppose $\varphi$ is a formula in the language $L^*(P)$. If $\varphi \in \Delta$ for every $\Delta \in \mathcal{G}$ with $\Gamma\mathcal{R}\Delta$, then $r{:}\varphi \in \Gamma$ for some pseudo-closed proof term $r$.*

**Proof** Suppose $r{:}\varphi \notin \Gamma$ for every pseudo-closed proof term $r$ in $L^*(P)$. The key item to show is that $\Gamma^\sharp \cup \{\neg\varphi\}$ is consistent. For then one can extend it to a world-like set $\Delta$; and then $\Gamma\mathcal{R}\Delta$, and $\neg\varphi \in \Delta$, which contradicts the hypothesis. So I now concentrate on showing that $\Gamma^\sharp \cup \{\neg\varphi\}$ is consistent.

Suppose $\Gamma^\sharp \cup \{\neg\varphi\}$ is not consistent. Then for some $Y_1, \ldots, Y_k \in \Gamma^\sharp$, there is a proof in $L^*(P)$ of $(Y_1 \wedge \ldots \wedge Y_k \wedge \neg\varphi) \supset \bot$, and hence there is also a proof of $(Y_1 \supset (Y_2 \supset \ldots (Y_k \supset \varphi)\ldots))$. For each $i = 1, \ldots, k$, since $Y_i \in \Gamma^\sharp$, there is some pseudo-closed proof term $s_i$ such that $s_i{:}Y_i \in \Gamma$. Using Proposition 3.4, there is a pseudo-closed proof term $p$ such that in $L^*(P)$ one can prove $p{:}(Y_1 \supset (Y_2 \supset \ldots (Y_k \supset \varphi)\ldots))$. Then repeated use of axiom 2 allows us to prove $(s_1{:}Y_1 \wedge \ldots \wedge s_k{:}Y_k) \supset (p \cdot s_1 \cdots s_k){:}\varphi$. Hence $(p \cdot s_1 \cdots s_k){:}\varphi \in \Gamma$, but this contradicts the original assumption that $r{:}\varphi \notin \Gamma$ for each pseudo-closed proof term $r$ of $L^*(P)$. ■

A domain function $\mathcal{D}$, is needed next, and this is simple. Suppose $\Gamma \in \mathcal{G}$ is $P$-world-like—recall, $P$ is uniquely determined by $\Gamma$. Define $\mathcal{D}(\Gamma)$ to be the set of all *pseudo-closed proof terms* in the language $L^*(P)$. (Remember, the occurrence of $x$ in $(t \,\forall\, x)$ is considered to be bound.) It is easy to see that we have monotonicity: $\Gamma\mathcal{R}\Delta$ implies $\mathcal{D}(\Gamma) \subseteq \mathcal{D}(\Delta)$. Note that the frame domain is the set of all pseudo-closed proof terms in the language $L^*$.

Next an interpretation function $\mathcal{I}$, is defined. For the function symbol $+$, define $+^{\mathcal{I}}$ to be the function that maps the pseudo-closed $L^*$ proof terms $t$ and $u$ to the pseudo-closed proof term $(t+u)$. And similarly for the function symbols $\cdot$, $!$, and the primitive function symbols. Thus far we have a kind of Herbrand interpretation. Next is the definition of $\forall^{\mathcal{I}}$, which requires more machinery.

Let $t$ be an arbitrary term of $L^*$ in which all the free variables are parameters except possibly for $x$. Define a mapping from pseudo-closed proof terms to pseudo-closed proof terms, denoted by $\langle\lambda^* x.t\rangle$, as follows. For a pseudo-closed proof term $r$, $\langle\lambda^* x.t\rangle(r)$ is the result of substituting occurrences of $r$ for free occurrences of $x$ in $t$. Notice that since $r$ is pseudo-closed, its only free variables are parameters, and parameters are never quantified, so this substitution is free for $x$ in $t$. Now, for functions of the form $\langle\lambda^* x.t\rangle$, set $\forall^{\mathcal{I}}(\langle\lambda^* x.t\rangle)$ to be the pseudo-closed proof term $(t \forall x)$. For functions not of this form, $\forall^{\mathcal{I}}$ is defined arbitrarily.

The model being constructed is a kind of Herbrand model, and so there is a close connection between valuations and substitutions. I introduce some formal machinery so that we can make use of this connection more easily. A *substitution* is a mapping from proof variables *other than parameters* to proof terms (the domain is not required to be finite). The set of proof terms is different between $L^{\mathsf{QLP}}$ and $L^*$; I'll allow proof terms from $L^*$ (in fact, I'm only interested in pseudo-closed ones). The symbol $\sigma$, with or without subscripts, will be used for substitutions. The result of applying the substitution $\sigma$ to the $\mathsf{QLP}$ formula $Z$ will be denoted by $Z\sigma$. Similarly $t\sigma$ is the result of applying $\sigma$ to the proof term $t$. It is, of course, assumed that substitutions only replace free occurrences of variables. As usual, a *valuation* in the structure $\langle\mathcal{G},\mathcal{R},\mathcal{D},\mathcal{I}\rangle$ is a mapping from proof variables to members of $\overline{\mathcal{D}}$, but for the canonical model being constructed, these domain members are pseudo-closed proof terms of $L^*$. Then one can naturally associate a substitution with each valuation in the obvious way: for a valuation $v$, denote the corresponding substitution by $\sigma_v$, where $\sigma_v$ is the substitution that replaces a variable $x$ by the pseudo-closed proof term $v(x)$.

**Lemma 6.6** *Let $t$ be any proof term in the language $L^*$, and let $v$ be any valuation in $\overline{\mathcal{D}}$, which is the collection of pseudo-closed proof terms. Then $t^v = t\sigma_v$.*

**Proof** The argument is by induction on the complexity of $t$. Most of the cases are straightforward, so I'll concentrate on the one that isn't. Suppose the result is known for $t$, and we wish to show that $(t \forall x)^v = (t \forall x)\sigma_v$.

By Definition 4.1, $(t \forall x)^v = \forall^{\mathcal{I}}(\langle\lambda x.t\rangle^v)$. Let us write $\sigma_v\backslash x$ for the substitution that is like $\sigma_v$ except that it leaves $x$ unchanged. Then $(t \forall x)\sigma_v = (t(\sigma_v\backslash x) \forall x)$, since $x$ is bound in $(t \forall x)$. And by the definition of $\forall^{\mathcal{I}}$, $(t(\sigma_v\backslash x) \forall x) = \forall^{\mathcal{I}}(\langle\lambda^* x.t\rangle(\sigma_v\backslash x))$. So it is enough to show that $\langle\lambda x.t\rangle^v = \langle\lambda^* x.t\rangle(\sigma_v\backslash x)$.

Let $r \in \overline{\mathcal{D}}$ be an arbitrary pseudo-closed term. By Definition 4.1 again, $\langle\lambda x.t\rangle^v(r) = t^w$ where $w$ is the $x$-variant of $v$ such that $w(x) = r$. By the induction hypothesis, $t^w = t\sigma_w$. And it is easy to see that $\langle\lambda^* x.t(\sigma_v\backslash x)\rangle(r) = t\sigma_w$, completing the argument. ■

Now to define an evidence function $\mathcal{E}$. For $\Gamma \in \mathcal{G}$, for $r \in \mathcal{D}(\Gamma)$, and for a valuation $v$, define $\mathcal{E}(\Gamma, r, v)$ to be the set of all formulas $X$ of $L^{\mathsf{QLP}}$ such that $r{:}(X\sigma_v) \in \Gamma$. If $r \notin \mathcal{D}(\Gamma)$, $r$ is not in $L^*(P)$ where $\Gamma$ is $P$-world-like, and it follows that $\mathcal{E}(\Gamma, r, v)$ is empty. Note that, in any case, members of $\mathcal{E}(\Gamma, r, v)$ are formulas in $L^{\mathsf{QLP}}$, they do not contain parameters.

There are seven conditions required for an evidence function. Condition 1 has been discussed. For condition 2, monotonicity, suppose $\Gamma\mathcal{R}\Delta$, and also suppose $X \in \mathcal{E}(\Gamma, r, v)$, which means that $r{:}(X\sigma_v) \in \Gamma$. Since members of $\mathcal{G}$ contain all axioms (in the appropriate language), $r{:}(X\sigma_v) \supset\ !r{:}r{:}(X\sigma_v)$ is in $\Gamma$, and since members of $\mathcal{G}$ are closed under modus ponens because of maximal consistency, $!r{:}r{:}(X\sigma_v) \in \Gamma$. Then $r{:}(X\sigma_v) \in \Gamma^\sharp \subseteq \Delta$, so $X \in \mathcal{E}(\Delta, r, v)$. Conditions 3, 4, and 5 on evidence functions are established similarly. Likewise condition 7 is straightforward.

Evidence function condition 6 needs a serious argument. Let $t$ be a proof term of $L^{\mathsf{QLP}}$, and suppose that for every $\Delta \in \mathcal{G}$ with $\Gamma\mathcal{R}\Delta$, and for every $r \in \mathcal{D}(\Delta)$, $X \in \mathcal{E}(\Delta, \langle\lambda x.t\rangle^v(r), v\binom{x}{r})$. We must show that $(\forall x)X \in \mathcal{E}(\Gamma, (t \forall x)^v, v)$. Our hypothesis amounts to saying that for every

$\Delta \in \mathcal{G}$ that is accessible from $\Gamma$, $\langle\lambda x.t\rangle^v(r){:}X\sigma_{v\binom{x}{r}} \in \Delta$, for every $r \in \mathcal{D}(\Delta)$. Now, by definition, $\langle\lambda x.t\rangle^v(r) = t^{v\binom{x}{r}}$ and by Lemma 6.6, this is equal to $t\sigma_{v\binom{x}{r}}$. Thus our hypothesis amounts to $t\sigma_{v\binom{x}{r}}{:}X\sigma_{v\binom{x}{r}} \in \Delta$, or equivalently, that $(t{:}X)\sigma_{v\binom{x}{r}} \in \Gamma$, and this is for each $r \in \mathcal{D}(\Delta)$. It follows using the $\exists$-completeness (and the maximal consistency) of $\Delta$ that $[(\forall x)(t{:}X)]\sigma_v \in \Delta$. Since $\Delta$ was arbitrary, Lemma 6.5 says that for some pseudo-closed proof term $r$ we have $r{:}[(\forall x)t{:}X]\sigma_v \in \Gamma$. Since $r$ is pseudo-closed, and substitutions do not replace parameters, this is equivalent to $[r{:}(\forall x)(t{:}X)]\sigma_v \in \Gamma$, and it follows that $[(\exists y)y{:}(\forall x)(t{:}X)]\sigma_v \in \Gamma$, where $y$ is a new variable. Then by axiom 8, $[(t \curlyvee x){:}(\forall x)X]\sigma_v \in \Gamma$. This tells us $(\forall x)X \in \mathcal{E}(\Gamma, (t \curlyvee x)\sigma_v, v)$ or, appealing to Lemma 6.6 again, $(\forall x)X \in \mathcal{E}(\Gamma, (t \curlyvee x)^v, v)$, which is what we wanted.

We have now completed verification that $\mathcal{E}$ is an evidence function.

Finally, define a mapping $\mathcal{V}$: for a propositional letter $P$, set $\mathcal{V}(P) = \{\Gamma \in \mathcal{G} \mid P \in \Gamma\}$.

The definition of a model, $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E}, \mathcal{V}\rangle$, is now complete; it will be called the *canonical model*. So far the conditions have been checked to verify that it is a weak QLP model. It is easy to see that it meets primitive term specification $\mathcal{F}$.

**Lemma 6.7 (Truth Lemma)** *Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E}, \mathcal{V}\rangle$ be the canonical model. For each $\Gamma \in \mathcal{G}$, for each formula $X$ in the language $L^{\mathsf{QLP}}$, and for each valuation $v$, if $X$ is meaningful at $\Gamma$ with respect to $v$, then $\mathcal{M}, \Gamma \Vdash_v X$ iff $X\sigma_v \in \Gamma$.*

**Proof** As usual, the argument is by induction on the complexity of $X$.

1. The atomic case. Let $X$ be $P$, a propositional letter. This is meaningful at every member of $\mathcal{G}$ with respect to every valuation, and the conclusion is immediate, by definition of $\mathcal{V}$.

2. The propositional connective case. If $X$ is $Y \supset Z$, and the Lemma is known for $Y$ and $Z$, it follows immediately for $X$ using the familiar properties of maximally consistent sets.

3. The quantifier case. This case too follows the route of standard completeness arguments. It makes use of axiom 6 and the $\exists$-completeness of members of $\mathcal{G}$.

4. The explicit proof term case. Here the argument is much as in [10]. Suppose $X$ is $t{:}Y$, and the Lemma is known for $Y$. For one direction of the Lemma, assume that $(t{:}Y)\sigma_v$ is in $\Gamma$. that is, $(t\sigma_v){:}(Y\sigma_v) \in \Gamma$. Then, first of all, $Y\sigma_v \in \Gamma^\sharp$, hence $Y\sigma_v \in \Delta$ for every $\Delta \in \mathcal{G}$ such that $\Gamma\mathcal{R}\Delta$, and so by the induction hypothesis, $\mathcal{M}, \Delta \Vdash_v Y$. And second, $Y \in \mathcal{E}(\Gamma, t\sigma_v, v)$, or equivalently, $Y \in \mathcal{E}(\Gamma, t^v, v)$. From these two it follows that $\mathcal{M}, \Gamma \Vdash_v t{:}Y$.

   For the other direction, suppose that $(t:Y)\sigma_v$ is meaningful at $\Gamma$ with respect to $v$ but $(t{:}Y)\sigma_v \notin \Gamma$. That is, $(t\sigma_v){:}(Y\sigma_v) \notin \Gamma$. By definition of $\mathcal{E}$, it is the case that $Y \notin \mathcal{E}(\Gamma, t\sigma_v, v)$ or equivalently, $Y \notin \mathcal{E}(\Gamma, t^v, v)$, and it follows that $\mathcal{M}, \Gamma \not\Vdash_v t{:}Y$.

■

It still must be shown that $\mathcal{M}$ is a *strong* QLP model, but the work has already been done. It must be shown that, for $X$ meaningful at $\Gamma \in \mathcal{G}$ with respect to valuation $v$, if $\mathcal{M}, \Delta \Vdash_v X$ for all $\Delta \in \mathcal{G}$ with $\Gamma\mathcal{R}\Delta$, then $X \in \mathcal{E}(\Gamma, r, v)$ for some $r \in \mathcal{D}(\Gamma)$. By the Truth Lemma, the hypothesis is equivalent to saying $X\sigma_v \in \Delta$ for all accessible $\Delta$. And by the definition of the evidence function, the conclusion is equivalent to saying $r:(X\sigma_v) \in \Gamma$ for some $r \in \mathcal{D}(\Gamma)$. Stated in this form, the implication holds by Lemma 6.5. Thus we have a strong model.

In the usual way, the canonical model is a universal counter-model. Let $X$ be a formula in the language $L^{\mathsf{QLP}}$ that does not have an axiomatic QLP proof. Since we have both universal generalization and axiom 6, it can be assumed that $X$ is closed without loss of generality. $\{\neg X\}$ is consistent. It can be extended to a world-like set $\Gamma$, which will be a state in the canonical model. And the Truth Lemma tells us that $\mathcal{M}, \Gamma \not\Vdash_v X$, for every valuation $v$, and so $X$ is invalidated in a strong QLP model.

# 7 Relationship With S4

Following Gödel, [12], one might think of the $\Box$ operator of S4 as corresponding to the existence of a proof in some sense. Artemov has made this precise, [2], showing an embedding result between S4 and LP, under which $\Box$ occurrences are replaced with explicit proof terms. In QLP one can actually say a proof, or reason, exists, so one should expect that S4 embeds into QLP by turning $\Box$ into an existential quantifier. This is the case, and will be shown in this section.

Let $L^{\mathsf{S4}}$ be the language built up from the same propositional letters as $L^{\mathsf{QLP}}$, using $\supset$ and $\bot$, without proof terms or quantifiers, but with the additional formation rule: if $X$ is a formula, so is $\Box X$. That is, $L^{\mathsf{S4}}$ is a standard propositional modal language, whose propositional letters are the same as in $L^{\mathsf{QLP}}$. Now I define an embedding from $L^{\mathsf{S4}}$ into $L^{\mathsf{QLP}}$, mapping formula $X$ to formula $X^{\exists}$, as follows.

1. If $P$ is a propositional letter, $P^{\exists} = P$
2. $\bot^{\exists} = \bot$
3. $(X \supset Y)^{\exists} = (X^{\exists} \supset Y^{\exists})$
4. $(\Box X)^{\exists} = (\exists x)x{:}(X^{\exists})$

In the last clause above, the actual choice of variable is not important—I've standardized on $x$. Note that for every $X$ of $L^{\mathsf{S4}}$, $X^{\exists}$ is a closed formula of $L^{\mathsf{QLP}}$.

**Theorem 7.1** *For each formula $X$ of $L^{\mathsf{S4}}$, $X$ is a theorem of S4 if and only if $X^{\exists}$ is a theorem of QLP.*

The proof of this will occupy the rest of the section. I'll begin with the following direction.

**Lemma 7.2** *If $X$ is a theorem of S4, then $X^{\exists}$ is a theorem of QLP using any axiomatically appropriate primitive term specification.*

There are two ways of showing Lemma 7.2: proof theoretically and semantically. Each is straightforward, but each has interesting points.

**Proof** Proceeding proof theoretically first, I show by induction on proof length that theorems of S4 translate to theorems of QLP. If $A$ is an axiom of S4 (using a standard axiomatization), it is simple to show that $A^{\exists}$ is a theorem of QLP. Here is one case as an example. The axiom $\Box Y \supset \Box\Box Y$ translates to $(\exists x)x{:}X \supset (\exists x)x{:}(\exists x)x{:}X$, where $X$ is $Y^{\exists}$. Here is a sketch of a proof of this in QLP.

1. $x{:}X \supset !x{:}x{:}X$. This is an instance of axiom scheme 4.
2. $x{:}X \supset (\exists x)x{:}X$ is provable (it is the dual of an instance of axiom 6)
3. $p{:}(x{:}X \supset (\exists x)x{:}X)$ is provable, for some proof term $p$, by 2 and Proposition 3.4.

4. $p{:}(x{:}X \supset (\exists x)x{:}X) \supset (!x{:}x{:}X \supset (p\cdot !x){:}(\exists x)x{:}X)$, an instance of axiom scheme 2.

5. $!x{:}x{:}X \supset (p\cdot !x){:}(\exists x)x{:}X$, from 3 and 4 by modus ponens.

6. $!x{:}x{:}X \supset (\exists x)x{:}(\exists x)x{:}X$, from 5.

7. $x{:}X \supset (\exists x)x{:}(\exists x)x{:}X$ from 1 and 6.

8. $(\exists x)x{:}X \supset (\exists x)x{:}(\exists x)x{:}X$, from 7 using the derived universal generalization rule, and some standard quantifier manipulation.

With axioms out of the way, I turn to the rules of inference. Applications of modus ponens translate to applications of modus ponens. Here is an argument to cover the necessitation case. Suppose, in S4, we conclude $\Box Y$ from $Y$, and in QLP we have a proof of $Y^{\exists}$. Proceed as follows. Using Proposition 3.4, $p{:}Y^{\exists}$, for some proof term $p$. And from this one easily gets $(\exists x)x{:}Y^{\exists}$, which is $(\Box Y)^{\exists}$. ■

Lemma 7.2 has now been proved in one direction, proof theoretically. It is interesting to also give a semantic proof of the same result.

**Proof** Suppose $X$ is a formula of $L^{\mathsf{S4}}$ and $X^{\exists}$ is *not* a theorem of QLP; I'll show $X$ is not a theorem of S4.

If $X^{\exists}$ is not a theorem of QLP, using the completeness result of Section 6 there is a strong QLP model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E}, \mathcal{V} \rangle$ in which $X^{\exists}$ is not valid. I'll show how to convert that to an S4 model in which $X$ is not valid. (I assume the reader is familiar with S4 models.) Very simply: let $\mathcal{N} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V} \rangle$ be the S4 model with the same set of possible worlds $\mathcal{G}$ as $\mathcal{M}$, the same accessibility relation $\mathcal{R}$ as $\mathcal{M}$, and with propositional letters true at the same worlds as in $\mathcal{M}$, as given by the mapping $\mathcal{V}$. That is, $\mathcal{N}$ is like $\mathcal{M}$ with all the quantificational and proof term structure forgotten. Now I show by induction on complexity that for each formula $Y$ of $L^{\mathsf{S4}}$, and for each possible world of $\mathcal{N}$ (or equivalently, of $\mathcal{M}$), that $\mathcal{M}, \Gamma \Vdash_v Y^{\exists}$ iff $\mathcal{N}, \Gamma \Vdash Y$. (The notation $\mathcal{N}, \Gamma \Vdash Y$ means that $Y$ is true at world $\Gamma$, with truth evaluated in the usual S4 manner.) Note that since $Y^{\exists}$ is always a closed formula, the choice of valuation $v$ is irrelevant.

Establishing this equivalence is a straightforward induction. The only interesting case is where $Y$ is of the form $\Box Z$, and the result has been established for $Z$. Here is one direction.

1. $\mathcal{M}, \Gamma \Vdash_v (\Box Z)^{\exists}$ is our starting point

2. $\mathcal{M}, \Gamma \Vdash_v (\exists x)x{:}Z^{\exists}$, which is 1 in more detail

3. $\mathcal{M}, \Gamma \Vdash_w x{:}Z^{\exists}$, for some $x$-variant $w$ of $v$, where $w(x) \in \mathcal{D}(\Gamma)$

4. $Z^{\exists} \in \mathcal{E}(\Gamma, x^w, w)$ and $\mathcal{M}, \Delta \Vdash_w Z^{\exists}$ for every $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{R} \Delta$

5. $\mathcal{N}, \Delta \Vdash Z$ for every $\Delta \in \mathcal{G}$ with $\Gamma \mathcal{R} \Delta$, by the induction hypothesis

6. $\mathcal{N}, \Gamma \Vdash \Box Z$

The other direction is more interesting.

1. $\mathcal{N}, \Gamma \Vdash \Box Z$ is the starting assumption

2. $\mathcal{N}, \Delta \Vdash Z$ for every $\Delta \in \mathcal{G}$ with $\Gamma \mathcal{R} \Delta$

3. $\mathcal{M}, \Delta \Vdash_w Z^\exists$ for every $\Delta \in \mathcal{G}$ such that $\Gamma\mathcal{R}\Delta$, by the induction hypothesis, where $w$ is an arbitrary valuation—pick one $w$ and use it in what follows

4. $Z^\exists \in \mathcal{E}(\Gamma, r, w)$ for some $r \in \mathcal{D}(\Gamma)$, from 3, using the fact that $\mathcal{M}$, being a strong QLP model, meets the fully explanatory condition

5. Let $v$ be the $x$-variant of $w$ such that $v(x) = r$. Since $x$ is not free in $Z^\exists$, $Z^\exists \in \mathcal{E}(\Gamma, r, v)$ using condition 7 on evidence functions.

6. It follows that $\mathcal{M}, \Gamma \Vdash_v x{:}Z^\exists$, and hence $\mathcal{M}, \Gamma \Vdash_w (\exists x)x{:}Z^\exists$, that is, $\mathcal{M}, \Gamma \Vdash_w (\Box Z)^\exists$

Note the use of the fully explanatory condition. The proof that it held in the canonical model made use of Proposition 3.4, which played an essential role in the proof-theoretic argument for Lemma 7.2 given above. ■

One direction of Theorem 7.1 has been proved (twice). Now I turn to the other direction.

**Lemma 7.3** *If $X^\exists$ is a theorem of QLP then $X$ is a theorem of S4.*

**Proof** This time the only proof given is semantic. I show the contrapositive, so to begin suppose $X$ is not a theorem of S4. Then there is an S4 model, $\mathcal{N} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V} \rangle$, in which $X$ is not valid. $\mathcal{N}$ will be used to construct a QLP counter-model for $X^\exists$. Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E}, \mathcal{V} \rangle$ be defined as follows. $\mathcal{G}$ is the same set of possible worlds as in $\mathcal{N}$, and $\mathcal{R}$ is the same accessibility relation. Likewise $\mathcal{V}$ assigns the same worlds to propositional letters as in $\mathcal{N}$. Let $a$ be some arbitrary object, and set $\mathcal{D}(\Gamma) = \{a\}$ for every $\Gamma \in \mathcal{G}$. Then define an interpretation $\mathcal{I}$ in the only possible way. For each primitive proof term $f(x_1, \ldots, x_n)$ of $L^{\mathsf{QLP}}$, set $f^\mathcal{I}(a, \ldots, a) = a$. Also set $!^\mathcal{I}(a) = a$, $a +^\mathcal{I} a = a$, and $a \cdot^\mathcal{I} a = a$. Likewise set $\forall^\mathcal{I}$ to map every function on $\overline{\mathcal{D}} = \{a\}$ (there is only one) to $a$. Of course only one valuation is possible, mapping every variable to $a$—call it $v$. For an evidence function, set $\mathcal{E}(\Gamma, a, v)$ to be the entire set of formulas of $L^{\mathsf{QLP}}$, for every $\Gamma \in \mathcal{G}$.

The structure $\mathcal{M}$ has now been fully characterized. The claim is that it is a strong QLP model. Most of the conditions are straightforward. For instance, $\mathcal{M}$ is fully explanatory because it has been required that $\mathcal{E}$ include every formula at every world, and for every reason.

I now show that for every formula $Z$ of $L^{\mathsf{S4}}$,

$$\mathcal{N}, \Gamma \Vdash Z \Longleftrightarrow \mathcal{M}, \Gamma \Vdash_v Z^\exists$$

In this, $\Gamma$ is any world in $\mathcal{G}$, and $v$ is the unique valuation.

The proof is by induction on the complexity of $Z$. The atomic case is by definition. The cases for $\bot$ and $\supset$ are straightforward. Now suppose $Z$ is $\Box W$, and the result is known for $W$. And suppose $\mathcal{N}, \Gamma \Vdash \Box W$. Then for each $\Delta \in \mathcal{G}$ with $\Gamma\mathcal{R}\Delta$ we have $\mathcal{N}, \Delta \Vdash W$, so by the induction hypothesis, $\mathcal{M}, \Delta \Vdash_v W^\exists$. Since $W^\exists \in \mathcal{E}(\Gamma, a, v)$ we have $\mathcal{M}, \Gamma \Vdash_v x{:}W^\exists$ (recall, $v(x) = a$). But then $\mathcal{M}, \Gamma \Vdash_v (\exists x)x{:}W^\exists$, that is, $\mathcal{M}, \Gamma \Vdash_v (\Box W)^\exists$. The other direction is similar, and is omitted.

Since $\mathcal{N}$ is an S4 counter-model to $X$ it follows that $\mathcal{M}$ is a strong QLP counter-model to $X^\exists$, and so $X^\exists$ is not a theorem of QLP. ■

# 8 Relationship With LP

As generally formulated, LP uses a constant specification and not a primitive term specification. Since a constant specification is a particular kind of primitive term specification, we can confine things to that case, and establish a relationship between LP and QLP, allowing only constants as primitive terms. On the other hand, LP could also be formulated to allow more general primitive proof terms. If so, a connection between LP and QLP can also be established, using the generalized version of LP. The argument is essentially the same either way, so I'll ignore details of primitive term specification $\mathcal{F}$. Obviously QLP is an extension of LP. It would be disconcerting if adding quantificational machinery had an effect on the propositional part. In this section I show that does not happen: QLP is a conservative extension of the propositional logic LP.

**Proposition 8.1** *Let $X$ be a formula in the language $L^{\mathsf{LP}}$—equivalently, $X$ is a quantifier-free formula of $L^{\mathsf{QLP}}$. If $X$ is not a theorem of LP, then $X$ is not a theorem of QLP, using the same primitive term specification $\mathcal{F}$ for both logics.*

**Proof** In [10] a semantics for LP is given. It will be used here—the reader is referred to that paper for details and terminology.

Suppose $X$ is not a theorem of LP. Then it is invalidated in some weak LP model. Let us say $X$ is not valid in the weak LP model $\mathcal{N} = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V}\rangle$. I'll use this to define a weak QLP model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E}', \mathcal{V}\rangle$. Note that we already have part of the characterization of $\mathcal{M}$. The underlying frame is the same as in $\mathcal{N}$. Likewise we are using the same mapping $\mathcal{V}$ of propositional letters to sets of worlds. What remains to specify is $\mathcal{D}$, $\mathcal{I}$, and $\mathcal{E}'$.

For each $\Gamma \in \mathcal{G}$, take $\mathcal{D}(\Gamma)$ to be the set of all proof terms in the language $L^{\mathsf{QLP}}$ (thus a constant domain model is being created). Let $!^{\mathcal{I}}$ be the function that maps the proof term $t$ to the proof term $!t$. Let $+^{\mathcal{I}}$ be the operation that maps the proof terms $s$ and $t$ to the proof term $(s+t)$. And so on. Define $\forall^{\mathcal{I}}$ in any arbitrary way. And finally, we specify the evidence function $\mathcal{E}'$. For each $\Gamma \in \mathcal{G}$ and for each valuation $v$, proceed as follows. Recall, members of the domain are proof terms. Now if $t$ does not involve the symbol $\forall$, set $\mathcal{E}'(\Gamma, t, v) = \mathcal{E}(\Gamma, t)$; if $t$ does contain the symbol $\forall$, set $\mathcal{E}'(\Gamma, t, v)$ to be the entire set of formulas of $L^{\mathsf{QLP}}$.

The structure $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E}', \mathcal{V}\rangle$ meets the conditions for being a weak model. I'll check a few of the cases.

One condition on the evidence function is: $(X \supset Y) \in \mathcal{E}'(\Gamma, s, v)$ and $X \in \mathcal{E}'(\Gamma, t, v)$ implies $Y \in \mathcal{E}'(\Gamma, s{\cdot}t, v)$. If neither $s$ nor $t$ involves the symbol $\forall$, this condition reduces to $(X \supset Y) \in \mathcal{E}(\Gamma, s)$ and $X \in \mathcal{E}(\Gamma, t)$ implies $Y \in \mathcal{E}(\Gamma, s \cdot t)$, which holds because it is one of the conditions that must be met in a weak LP model as defined in [10]. If either $s$ or $t$ contains the symbol $\forall$, so does $s \cdot t$, and so $(X \supset Y) \in \mathcal{E}'(\Gamma, s, v)$ and $X \in \mathcal{E}'(\Gamma, t, v)$ implies $Y \in \mathcal{E}'(\Gamma, s \cdot t, v)$ is true since $\mathcal{E}'(\Gamma, s \cdot t, v)$ is the entire set of formulas.

Condition 6 on evidence functions, having to do with $\forall$, is trivially satisfied, because $\mathcal{E}'(\Gamma, (t \forall x), v)$ is the set of all formulas.

Now it is easy to check that for each formula $Z$ in the propositional language of LP, for each world $\Gamma \in \mathcal{G}$, and using the valuation $v$ such that $v(x) = x$,

$$\mathcal{M}, \Gamma \Vdash_v Z \Longleftrightarrow \mathcal{N}, \Gamma \Vdash Z$$

and this is sufficient to establish the QLP invalidity of $X$. ■

## 9 Conclusion

QLP seems to be a rich and expressive logic, with the flavor of LP, plus the ability to quantify over proofs or reasons. But there is more investigation to carry out. As a simple example, is there some cut-free Gentzen system or tableau system for QLP?

Moving outside the bounds considered in this paper, in [4] extensions of LP were considered, containing a primitive $\Box$ operator as well as the LP machinery. It would be interesting to see if there were some natural relationship between what was done there and QLP, which has its own way of representing $\Box$ using the existential quantifier. Also, in [3] LP machinery was combined with that of more conventional multi-agent logics of knowledge. Now suppose that, in addition, one also allows quantification over reasons, as is done here. With such machinery, interesting possibilities arise. We could, for instance, distinguish between agent $i$ knowing $X$ has a reason $K_i(\exists x)x{:}X$ and agent $i$ having a reason for $X$, $(\exists x)K_i x{:}X$, or even $K_i t{:}X$, for some explicit $t$. Or we could say of two agents, that $i$ has a reason for $X$, and $j$ knows this, without having to say that $j$ knows what the reason is, $(\exists x)K_i x{:}X \wedge K_j(\exists x)K_i x{:}X$. We might even be able to say that agent $j$ knows there is a reason for $X$, but does not know what it is; a first pass might be: $K_j(\exists x)x{:}X \wedge \neg(\exists x)K_j x{:}X$. But of course without a properly developed semantics, it is not really clear that any of these have the meanings we want. This must be left to future work.

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