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A SYMMETRIC APPROACH TO AXIOMATIZING QUANTIFIERS AND MODALITIES

1. INTRODUCTION

We present an axiomatization of several of the basic modal logics, with the idea of giving the two modal operators □ and ◇ equal weight as far as possible. Then we present a parallel axiomatization of classical quantification theory, working our way up through a sequence of rather curious subsystems. It will be clear at the end that the essential difference between quantifiers and modalities is amusing in a vacuous sort of way. Finally we sketch tableau proof systems for the various logics we have introduced along the way. Also, the "natural" model theory for the subsystems of quantification theory that come up is somewhat curious. In a sense, it amounts to a "stretching out" of the Henkin-style completeness proof, severing the maximal consistent part of the construction quite thoroughly from the part of the construction that takes care of existential-quantifier instances.

2. BACKGROUND

It is contrary to the spirit of what we are doing to take one modal operator as primitive and define the other, or, one quantifier as primitive and define the other. So, by the same token, we take as primitive all the standard propositional connectives too. Thus, we have available all of $\wedge$, $\vee$, $\sim$, $\supset$, □, ◇, $\forall$, $\exists$. We also take as primitive a truth constant $\top$ and a falsehood constant $\perp$.

For our treatment of propositional modal logic we assume we have a countable list of atomic formulas, and that the set of formulas is built up from them in the usual way. We will use the letters "X", "Y", etc., to denote such formulas.

For quantification theory, we assume we have a countable list of variables and also a disjoint countable list of parameters. We will use the letters "x", "y", etc., to denote variables, and "a", "b", etc., to denote parameters. Formulas are built up in the usual way, with the understanding that a sentence contains no free variables, though it may

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contain parameters. We follow the convention that if $\varphi(x)$ is a formula with only x free, then $\varphi(a)$ is the result of replacing all free occurrences of x by occurences of a in φ.

We will use Kripke's model theory for modal logics (and an analog for quantificational theories, to be described in Section 4). For us, a *Kripke model* is a quadruple $\langle \mathcal{G}, \Phi, \mathcal{R}, \Vdash \rangle$ where: (1) $\mathcal{G}$ is a nonempty set (of possible worlds); (2) Φ is a possible empty subset of $\mathcal{G}$ (of so-called queer worlds); (3) $\mathcal{R}$ is a relation on $\mathcal{G}$ (of accessibility); and (4) $\Vdash$ is a relation between members of $\mathcal{G}$ and formulas meeting the following conditions (we write $\Gamma \nVdash X$ as short for not $-\ \Gamma \Vdash X$):

for every $\Gamma \in \mathcal{G}$

(a) $\Gamma \Vdash \mathrm{T}$ and $\Gamma \nVdash \bot$;
(b) $\Gamma \Vdash (X \wedge Y)$ iff $\Gamma \Vdash X$ and $\Gamma \Vdash Y$;
(c) $\Gamma \Vdash (X \vee Y)$ iff $\Gamma \Vdash X$ or $\Gamma \Vdash Y$;
(d) $\Gamma \Vdash {\sim} X$ iff $\Gamma \nVdash X$;
(e) $\Gamma \Vdash (X \supset Y)$ iff $\Gamma \nVdash X$ or $\Gamma \Vdash Y$;

for every $\Gamma \in \Phi$

(f) $\Gamma \Vdash \Diamond X$ but $\Gamma \nVdash \Box X$;

for every $\Gamma \in \mathcal{G} - \Phi$

(g) $\Gamma \Vdash \Box X$ iff for every $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{R} \Delta$, $\Delta \Vdash X$;
(h) $\Gamma \Vdash \Diamond X$ iff for some $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{R} \Delta$, $\Delta \Vdash X$.

Thus, in a queer world, anything is possible, nothing is necessary. But otherwise, necessary truth means truth in all accessible worlds. A Kripke model in which $\Phi = \phi$ is called *normal*. See Kripke [2] and [3].

We also will be making much use of *uniform notation*, as introduced in Smullyan [6]. For this purpose, the nonatomic formulas of propositional logic are grouped into α-formulas (those that act conjuctively) and β-formulas (those that act disjunctively). These, and their *components*, α_1, α_2 and β_1, β_2, respectively, are presented in the following charts:

α	α_1	α_2
$(X \wedge Y)$	X	Y
${\sim}(X \vee Y)$	${\sim} X$	${\sim} Y$
${\sim}(X \supset Y)$	X	${\sim} Y$
${\sim}{\sim} X$	X	X

β	β_1	β_2
$(X \vee Y)$	X	Y
${\sim}(X \wedge Y)$	${\sim} X$	${\sim} Y$
$(X \supset Y)$	${\sim} X$	Y

Notice that (b)–(e) in the definition of a Kripke model above gives us

$$\Gamma \Vdash \alpha \text{ iff } \Gamma \Vdash \alpha_1 \text{ and } \Gamma \Vdash \alpha_2$$
$$\Gamma \Vdash \beta \text{ iff } \Gamma \Vdash \beta_1 \text{ or } \Gamma \Vdash \beta_2.$$

Indeed, this could be used instead of (b)–(e).

We extend this uniform notation to the modal operators (as in Fitting [1]) by defining the ν-formulas (necessaries) and π-formulas (possibles) and their components ν_0 and π_0, respectively, as follows:

ν	ν_0
$\Box X$	X
$\sim\Diamond X$	$\sim X$

π	π_0
$\Diamond X$	X
$\sim\Box X$	$\sim X$

Notice that (f)–(h) in the definition of a Kripke model above, taken together with (d), give us the following equivalent conditions:

for every $\Gamma \in \Phi$,

(f′) $\Gamma \Vdash \pi$ but $\Gamma \nVdash \nu$;

for every $\Gamma \in \mathcal{G} - \Phi$

(g′) $\Gamma \Vdash \nu$ iff for every $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{R} \Delta$, $\Delta \Vdash \nu_0$;

(h′) $\Gamma \Vdash \pi$ iff for some $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{R} \Delta$, $\Delta \Vdash \pi_0$.

Finally we present uniform notation for the quantifiers, as in Smullyan [6] or as more fully developed, Smullyan [7]. The quantified sentences are divided into γ-sentences (universals) and δ-sentences (existentials). This time, rather than components, we have *instances*, γ(a) and δ(a), one for each parameter a.

γ	$\gamma(a)$
$(\forall x)\ \phi(x)$	$\phi(a)$
$\sim(\exists x)\ \phi(x)$	$\sim\phi(a)$

δ	$\delta(a)$
$(\exists x)\ \phi(x)$	$\phi(a)$
$\sim(\forall x)\phi(x)$	$\sim\phi(a)$

The idea is, in a classical first-order model in which the domain consists of the set of parameters, and each parameter is interpreted as naming itself:

γ is true iff for all a, $\gamma(a)$ is true;

δ is true iff for some a, $\delta(a)$ is true.

So much for background.

3. MODAL LOGICS, AXIOMATICALLY

Let us assume that we have an axiomatization of the classical propositional calculus, with modus ponens as the only rule. We will build on that in our introduction of rules and axioms pertinent to the modal operators.

Modal logics are often formulated with a rule of necessitation, which we could give as

$$\frac{\nu_0}{\nu}.$$

But this gives a central role to the ν-formulas and, for no reason other than autocratic whim, we want to develop modal logic as far as possible giving equal weight to both ν-and π-formulas, giving to both necessaries and possibles a fair share. After a certain amount of experimentation we hit on the following curious rule:

$$(M) \qquad \frac{\pi_0 \vee \nu_0}{\pi \vee \nu}$$

(we use M for modalization). As a matter of fact, rule M is a correct rule of inference, as the following argument shows.

Suppose $\langle \mathcal{G}, \Phi, \mathcal{R}, \Vdash \rangle$ is a Kripke model, and $\pi_0 \vee \nu_0$ is valid in it (holds at every possible world). Let $\Gamma \in \mathcal{G}$; we show $\Gamma \Vdash \pi \vee \nu$. Well, if $\Gamma \in \Phi$, that is, if Γ is queer, then $\Gamma \Vdash \pi$; so, trivially, $\Gamma \Vdash \pi \vee \nu$. Otherwise, $\Gamma \in \mathcal{G} - \Phi$. Now suppose $\Gamma \nVdash \pi$. Then it must be that in every Δ such that $\Gamma \mathcal{R} \Delta$, we have $\Delta \nVdash \pi_0$. But $\pi_0 \vee \nu_0$ is assumed valid in our model, so it holds at every such Δ. Thus for every Δ such that $\Gamma \mathcal{R} \Delta$, $\Delta \Vdash \nu_0$; hence $\Gamma \Vdash \nu$. Thus again $\Gamma \Vdash \pi \vee \nu$.

Rule M (which is actually four rules when translated out of uniform notation) is quite a useful rule. It is rather easy to show that, by using it, one may derive the usual interdefinitions of the modal operators (as mutual equivalences). Also one may show the following are derived rules:

$$\frac{X \supset Y}{\Box X \supset \Box Y} \qquad \frac{X \supset Y}{\Diamond X \supset \Diamond Y}.$$

Then one may show, in the usual way, that replacement of proved equivalences holds as a derived rule:

$$\frac{X \equiv X'}{Z \equiv Z'}$$

where Z' results from Z by replacing some occurrences of X in Z by occurrences of X'. Here we have used $\equiv$ as an abbreviation for mutual implication. (Actually a stronger version may be shown, concerning "semisubstitutivity" of implication, in which we must take into account the positiveness and negativeness of occurrences as well. The details needn't concern us here.)

Now go back and look again at the justification we gave for rule M. In the argument that $\pi \vee \nu$ held at the nonqueer world Γ we never needed that $\pi_0 \vee \nu_0$ held at *every* world of the model; we only needed that it held at every world accessible from Γ. But to say that $\pi_0 \vee \nu_0$ holds at every world accessible from Γ is to say that $\Box(\pi_0 \vee \nu_0)$ holds at Γ. Thus, the same argument also shows the validity in all Kripke models of the schema

(M1) $\quad \Box[\pi_0 \vee \nu_0] \supset [\pi \vee \nu]$.

Let us add it as an axiom schema then.

If we do so, it is not hard to show that we have a complete axiomatic counterpart of the Kripke model theory as given in section 2. That is, X is provable in the axiomatic system just described iff X holds at every world of every Kripke model. To show this, one may show completeness directly, using the now-common Lindenbaum-style construction, or one may show this axiom system is of equal strength with one standard in the literature, and rely on known completeness results. We skip the arguments.

The logic axiomatically characterized thus far is called C in Segerberg [5]. It is the smallest *regular* modal logic (see Segerberg [5]).

Now, in the model theory for C, one can have queer worlds in which everything is possible but nothing is necessary. Also there are no special conditions placed on the accessibility relation $\mathcal{R}$, so there can be "dead-end" worlds, worlds from which no world is accessible. In such a world, everything is necessary, nothing is possible. So, our next item of business is to rule out such strange worlds.

Worlds in which nothing is necessary may be eliminated by postulating that something is necessary. Let us, then, add the axiom

(M2) $\Box \mathrm{T}$.

The model theory appropriate to this (with respect to which one can prove completeness) is all Kripke models in which there are no queer worlds, that is, all normal Kripke models. The logic axiomatized thus far is the smallest *normal* logic, and is usually called K (see Segerberg [5]).

Next, worlds in which nothing is possible may be eliminated by postulating that something is possible. We take as an axiom

(M3) $\Diamond \mathrm{T}$.

The model theory appropriate to this is all normal Kripke models in which every world has some world accessible to it. The logic is generally called D. (Again, see Segerberg [5]).

Finally, we might want to restrict our attention to models in which each (normal) world is accessible to itself (in which the accessibility relation is reflexive). To do this one adds either (or both) of

(M4) $\nu \supset \nu_0$
$\pi_0 \supset \pi$

The logic thus characterized is T.

Note that $\mathrm{T} \supset \Diamond \mathrm{T}$ is an instance of the second of these schemas, and since T is a tautology, $\Diamond \mathrm{T}$ follows by modus ponens. Thus with M4 added, M3 becomes redundant.

REMARKS. One goes from C to K by adding $\Box \mathrm{T}$ as an axiom. It is of some interest to consider a kind of halfway point, where instead of adding $\Box \mathrm{T}$ as an *assumed truth*, we take it as a *hypothesis*. That is, form the set S of formulas X such that $\Box \mathrm{T} \supset X$ is provable in C. This set S is, itself, a (rather strangely defined) modal logic, intermediate between C and K. It is closed under modus ponens, but not under rule M. Rather, it is closed under the weaker rule.

$$\frac{\Box(\pi_0 \vee \nu_0)}{\Box(\pi \vee \nu)},$$

which is equivalent to Becker's rule. It is, in fact, the logic axiomatized in Lemmon [4] as P2, but without his axiom $\Box X \supset X$ (our M4).

If we strengthen things a bit, by considering those X such that $\Box T \supset X$ is provable in C plus axiom M4, one gets the Lewis system S2. (Again, see Segerberg [5], chapter four.)

One can also play similar games with axiom M3 ($\Diamond T$) to produce interesting logics. We know very little about them.

4. QUANTIFIED LOGICS, AXIOMATICALLY

There is an obvious analogy (of sorts) between the modal operators and quantifiers. What we do in this section is to parallel the development of section 3, substituting quantifiers for the modal operators to see how far the analogy extends when things are done the way we did. The idea is simple: $(\forall x)$ may behave like $\Box$, $(\exists x)$ like $\Diamond$, γ like ν and δ like π. This may be so; we will see.

The language now is first order. Once again we assume a propositional-logic base with modus ponens as the sole rule.

First, the analog of rule M is

$$\text{rule (Q)} \quad \frac{\delta(a) \vee \gamma(a)}{\delta \vee \gamma}$$

And, as a matter of fact, this is a correct rule of inference in classical first-order logic. This argument is left to the reader.

Using rule Q, one may show analogs of the results listed in section 3 based on rule M. Thus, one may derive the usual inter definability of the quantifiers (again, as mutual implication), and one may show that replacement of proved equivalences holds as a derived rule.

Now, if you actually thought through the "justification" of rule Q, almost certainly you also showed the validity, in all first-order models, of the schema

$$\text{(Q1)} \qquad (\forall x)[\delta(x) \vee \gamma(x)] \supset [\delta \vee \gamma].$$

So let us add it as an axiom schema. We might, by analogy, call the resulting logic QC. It thus has rule Q and schema Q1.

The following is a reasonable question: What is an adequate model theory for QC, one with respect to which completeness can be shown? Well, the following rather curious one will do. We simply translate the corresponding modal model theory, making suitable adjustments to take care of the fact that quantified sentences have many instances, but modalized formulas have single components. We have chosen, for simplicity, to leave out any mention of the notion of an *interpretation* in a model. A more elaborate treatment would have to include it, but the following is enough for our purposes.

A *model* is a quintuple $\langle \mathcal{G}, \Phi, \mathcal{P}, \mathcal{R}, \Vdash \rangle$ where: (1) $\mathcal{G}$ is a nonempty set (of possible worlds); (2) $\Phi \subseteq \mathcal{G}$; (3) $\mathcal{P}$ is a mapping from members of $\mathcal{G}$ to nonempty sets of parameters; (4) $\mathcal{R}$ is a relation on $\mathcal{G}$; and (5) $\Vdash$ is a relation between members of $\mathcal{G}$ and sentences such that

for every $\Gamma \in \mathcal{G}$,

conditions (a)–(e) as in section 1;

for every $\Gamma \in \Phi$,

(f) $\Gamma \Vdash (\exists x)\varphi(x)$ but $\Gamma \nVdash (\forall x)\varphi(x)$;

for every $\Gamma \in \mathcal{G} - \Phi$

(g) $\Gamma \Vdash (\forall x)\varphi(x)$ iff for every $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{R} \Delta$, and for every $a \in \mathcal{P}(\Delta)$, $\Delta \Vdash \varphi(a)$

(h) $\Gamma \Vdash (\exists x)\varphi(x)$ iff for some $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{R} \Delta$, and for some $a \in \mathcal{P}(\Delta)$, $\Delta \Vdash \varphi(a)$.

Now, a sentence X is a theorem of the logic QC iff X holds at every world of every such model. We leave the correctness half to the reader. Note that if $\varphi(a)$ is provable in QC, so is the parameter variant $\varphi(b)$. This will be of use in proving correctness. And we briefly sketch the completeness half in the next section.

We note that we could restrict models so that, for each world Γ, $\mathcal{P}(\Gamma)$ is a singleton. It makes no difference. Now we can continue with our development, paralleling that of section 3.

In the quantifier models above there can be "queer" worlds (members of φ) in which everything exists, and hence no universal sentences hold. Such anomalies can be eliminated by adding the postulate

(Q2) $(\forall x)\mathsf{T}$.

Doing so gives us a logic we may call QK. An adequate model theory for it is one that consists of all models of the sort described above, but with Φ always empty, that is no "queer" worlds.

Next, there may still be worlds from which no world is accessible. In such a world, every universal sentence holds, but no existential. They behave rather like empty-domain models of first-order logic. They may be ruled out by adding the axiom

(Q3) $(\exists x)\top$.

Let us call the resulting logic QD. Its model theory is that of QK with the additional requirement that for every world Γ there must be some world Δ such that $\Gamma\mathcal{R}\Delta$.

Finally (?) we may add an analog to M4, namely

(Q4) $\gamma \supset \gamma(a)$ or
$\delta(a) \supset \delta$ (or both).

As might be expected, Q3 then becomes redundant. We can call the logic thus axiomatized QT. Its model theory is that of QD with the restriction that the accessibility relation $\mathcal{R}$ be reflexive.

We have reached the end of our parallel development (obviously, since we have matched everything we did in section 3). But we do *not* yet have the usual first-order logic. One doesn't want classical first-order models with lots of possible worlds in them: a classical model should be a one-world model.

Now, to only consider one-world Kripke modal models is to trivialize modal logic; it renders the modal operators useless. Necessary truth becomes the same thing as truth. That is, $A \supset \Box A$ is valid in all one-world models. Of course, this is not desirable. Modal operators are supposed to do something; they are supposed to have an effect; they ought not be vacuous.

Well, it is precisely at this point that modal operators and quantifiers diverge. Quantifiers can be vacuous. Let φ be a sentence (hence with no free variables). Then $(\forall x)\varphi$ ought to mean nothing more than φ itself. So our final quantificational axiom schema is, simply,

(Q5) $\varphi \supset (\forall x)\varphi$, where φ is a sentence.

When this is added, conventional classical first-order logic is the result.

5. COMPLETENESS, HOW PROVED

In a sense, the proof of the completeness of the quantifier system QC with respect to the model theory presented in the previous section is a "stretching out" of the usual Henkin completeness proof for first-order logic. Let us sketch what we mean by this.

Recall that the usual Henkin argument runs as follows. Take a consistent set, extend it to a maximal consistent one, then throw in "witnessess" for the existential quantifiers. But that will destroy maximality, so extend to a maximal consistent set again. But that may add new existential quantifiers, so add new witnesses, and so on. One sequentially alternates a maximal consistent construction with an existential-instantiation construction (and then takes the limit). It will be seen shortly that the completeness proof for QC really amounts to a separation of these two constructions.

First we say what consistency means. A set S is *inconsistent* if, for some finite subset $\{A_1, \ldots, A_n\} \subseteq S$, $(A_1 \wedge \ldots \wedge A_n) \supset \bot$ is a theorem of QC. A set S is *consistent* if it is not inconsistent. We adopt this definition in part because the deduction theorem is not available for QC. Indeed, we do not get the deduction theorem for any of the proper subsystems of quantificational logic presented in section 4.

Next, partition the set of parameters into countably many disjoint sets, each with countably many members: $P_1, P_2, P_3, \ldots$.

Let F be a sentence not provable in QC; we produce a countermodel for F. Without loss of generality we assume any parameters of F are in P_1.

First, extend $\{\sim F\}$ to a maximal consistent subset of the set of all sentences with parameters in P_1; call this Γ_F. And set $\mathcal{P}(\Gamma_F) = P_1$.

Next if there are no γ sentences in Γ_F, do nothing (but call Γ_F "queer"). Likewise if there are no δ sentences in Γ_F. do nothing. Otherwise, using rule (Q), it is not hard to show that if $\gamma_1, \gamma_2, \gamma_3, \ldots$ and δ are in Γ_F. and $a \in P_2$ (hence is new to Γ_F) that $\gamma_1(a)$, $\gamma_2(a)$, $\gamma_3(a), \ldots, \delta(a)$ is consistent. (Using Q1 it can be shown that any finite number of γ sentences is equivalent to a single one; this is needed here). This in turn can be extended to a maximal consistent subset of the set of all sentences with parameters in $P_1 \cup \{a\}$. Call it Γ_a. And set $\mathcal{P}(\Gamma_a) = \{a\}$, and $\Gamma_F \mathcal{R} \Gamma_a$. Do this sort of thing for each δ sentence in Γ_F, producing a whole batch of sets $\Gamma_a, \Gamma_b, \Gamma_c, \ldots$, all accessible from Γ_F under $\mathcal{R}$. Now repeat the process with each of these new maximal consistent sets, then with the sets that will arise from that, and so on.

Finally, define $\Vdash$ between the various sets Γ that arise this way, and sentences, as follows. If Γ is "queer", $\Gamma \Vdash \delta$ for all δ. If A is atomic, $\Gamma \Vdash A$ iff $A \in \Gamma$. And finally, the conditions of section 4 can be turned around to say how to define $\Vdash$ (inductively on formula degree) now that we have covered the atomic case. When this is done, a model results, for which $X \in \Gamma \Rightarrow \Gamma \Vdash X$ (but not conversely). And it will be a countermodel to F, since $\sim F \in \Gamma_F$.

Notice that in the proof just sketched, the maximal consistent construction gives the worlds, while the existential instantiation moves things from one world to another. This is what we meant by "stretching out" the Henkin construction.

Adding axioms (Q2)–(Q4) modifies the construction in obvious ways. We leave this to the reader.

Now we can ask, What is the effect on this construction of also imposing that final axiom schema (Q5), $\varphi \supset (\forall x)\varphi$? Very simply, it makes things cumulative. Notice that, in our model, if $\Gamma\mathscr{R}\Delta$, then moving from Γ to Δ, in effect, drops one quantifier from each sentence. But $\varphi \supset (\forall x)\varphi$ allows us to add one quantifier, so–the effect is neutralized. Briefly, if we assume (Q5), then if $\Gamma \Vdash \varphi$ and $\Gamma\mathscr{R}\Delta$, then $\Delta \Vdash \varphi$. For, if $\Gamma \Vdash \varphi$, since also $\Gamma \Vdash \varphi \supset (\forall x)\varphi$, we must have $\Gamma \Vdash (\forall x)\varphi$. And since $\Gamma\mathscr{R}\Delta$, then $\Delta \Vdash$ some-instance-of-φ. But since the quantifier was vacuous, this means $\Delta \Vdash \varphi$. Now, that things are cumulative if (Q5) is imposed means the limit (=chain-union) part of Henkin's proof can be carried out. And thus a conventional classical model results.

This sketch must suffice. Details, though slightly devious, are far from devastating.

1. SEMANTIC TABLEAUX

We show how the tableau system of Smullyan [7] for propositional logic may be extended to handle the logics discussed in section 4. We begin with a brief sketch of the system that suffices for propositional logic.

First, proofs are in tree form (written branching downward). There are two *branch-extension rules*:

$$\begin{array}{c}\alpha\\ \hline \alpha_1\\ \alpha_2\end{array} \qquad\qquad \begin{array}{c}\beta\\ \swarrow \quad \searrow\\ \beta_1 \qquad \beta_2\end{array}$$

(If α occurs on a branch, α_1 and α_2 may be added to the end of the branch. If β occurs on a branch, the end of the branch may be split, and β_1 added to the end of one fork, β_2 to the end of the other.)

A branch is called *closed* if it contains A and $\sim A$ for some formula A, or if it contains $\perp$, or if it contains $\sim$T. A tree is called closed if every branch is closed. A closed tree with $\sim X$ at the origin is, by definition, a *proof* of X.

We begin by adding to the above a tableau rule to give the modal logic C. In words, the rule is as follows. If, on a branch, there are ν-formulas, and there is a π-formula, then that π-formula may be replaced by π_0, all the ν-formulas by the corresponding ν_0-formulas, and all other formulas deleted.

We schematize this as follows. First, if S is a set of formulas, define

$$S\# = \{\nu_0 \mid \nu \in S\}.$$

Then the rule is

$$\frac{S,\ \pi}{S\#,\ \pi_0} \quad \text{(provided } S\# \neq \phi\text{)},$$

where this is to be interpreted as follows: if $S \cup \{\pi\}$ is the set of formulas on a branch, it may be replaced by $S\# \cup \{\pi_0\}$ (provided $S\#$ is not empty).

EXAMPLE. We show $\Box X \supset \sim\Diamond\sim X$ is provable using this rule.

The proof begins by putting $\sim(\Box X \supset \sim\Diamond\sim X)$ at the origin, then two α-rule applications produce the following one-branch tree:

$$\begin{array}{l} \sim(\Box X \supset \sim\Diamond\sim X) \\ \Box X \\ \sim\sim\Diamond\sim X \\ \Diamond\sim X. \end{array}$$

Now take S to consist of the first three formulas, and π to be $\Diamond\sim X$. Then $S\# = \{X\}$, which is not empty, so the rule says the set of formulas on the branch may be replaced by $S\# \cup \{\pi_0\}$, namely

$$\begin{array}{l} X \\ \sim X \end{array}$$

and this is closed!

REMARK. Because of the way trees are written, an occurrence of a formula may be common to several branches, but we may wish to modify (or delete) it using the above rule on only one branch. Then, simply, first add new occurrences of the formula at the ends of all the branches that are not to be modified, then use the above rule on the branch to be worked on.

Now the other modal logics can be dealt with easily.

The logic K was axiomatized by adding $\Box\mathrm{T}$. In effect this says there are always ν-formulas available, hence $S\#$ can always be considered to be nonempty. And, in fact, the appropriate tableau system for K is the one above without the provision that $S\#$ be nonempty.

The logic D had $\Diamond\mathrm{T}$ as an axiom. In effect, this says there is always a π-formula around, so an explicit occurrence of π need not be present to apply the tableau rule. Properly speaking, a tableau system for D results from that of K by adding the additional rule:

$$\frac{S}{S\#}.$$

Finally the logic T had as an axiom schema $\nu \supset \nu_0$. Well, simply add the tableau rule

$$\frac{\nu}{\nu_0}$$

(it can easily be shown that the D-rule above is redundant).

For quantifiers we proceed analogously of course. Thus, for a parameter a, and a set S of first-order sentences, let

$$S(a) = \{\gamma(a) \mid \gamma \in S\}.$$

Then a tableau system for QC is the Smullyan propositional system plus the rule (to be read in a similar fashion to the one for C above)

$$\frac{S,\ \delta}{S(a),\ \delta(a)}$$

provided

(1) a is new to the branch;
(2) $S(a) \neq \phi$.

For QK, drop the requirement that $S(a)$ be nonempty.
For QD, add the rule

$$\frac{S}{S(a)}.$$

provided a is new to the branch.
For QT, add the rule

$$\frac{\gamma}{\gamma(a)}$$

for any parameter a.
(This makes the QD rule redundant).

And finally, for first-order logic proper we want the "cumulativeness' that $\varphi \supset (\forall x)\varphi$ brought; that is, as we go on in a tableau construction, sentences should never be deleted. Well, we could replace the QK rule by

$$\frac{S, \delta}{S, S(a), \delta(a)}$$

provided a is new to the branch.

The only other quantifier rule now is

$$\frac{\gamma}{\gamma(a)}.$$

It is not hard to see that these are equivalent to the simpler set

$$\frac{\delta}{\delta(a)} \text{ a new } \qquad \frac{\gamma}{\gamma(a)},$$

and we have exactly the first-order system of Smullyan [7].

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