First-Order Logic
Problem Solutions

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Abstract

These are solutions to selected problems in *First-Order Logic and Automated Theorem Proving.*

Exercise 2.3.1 Show that all binary connectives can be defined using $\neg$ and any one of the Primary Connectives.

For this, one simply does a case-by-case analysis. I'll show $\uparrow$ can be defined using negation and each of the Primary Connectives. Then I'll appeal to Exercise 2.3.2, which says $\uparrow$ is sufficient to define all binary connectives. The formula $P \uparrow Q$ is equivalent to each of the following: $\neg(P \wedge Q)$, $\neg P \vee \neg Q$, $P \supset \neg Q$, $\neg P \subset Q$, $\neg(\neg P \downarrow \neg Q)$, $\neg(P \not\supset \neg Q)$, $\neg(\neg P \not\subset Q)$.

Exercise 2.3.2 Show that all binary connectives can be defined using either $\uparrow$ or $\downarrow$ alone.

I'll just discuss $\uparrow$; the other is similar. Here are the necessary equivalents. First, $\neg P \equiv (P \uparrow P)$, so we can make use of negation. Then $(P \wedge Q) \equiv \neg(P \uparrow Q)$. Since we have $\wedge$ and $\neg$, $(P \vee Q) \equiv \neg(\neg P \wedge \neg Q)$, which can be translated back into just $\uparrow$. And so on. I'll omit the rest, which are straightforward once one has $\neg$, $\wedge$, and $\vee$.

Exercise 2.3.3 Show that no Primary Connective can be defined using $\neg$ and the Secondary Connectives.

This one is hard. Perhaps I've forgotten my original argument. But here is a solution. First, suppose we have *only* $\equiv$ to work with. What truth functions can be defined, starting with P and Q? It is not hard to show that $\equiv$ is commutative and associative. That is, each of the following is a tautology:

$$(X \equiv Y) \equiv (Y \equiv X)$$
$$((X \equiv Y) \equiv Z) \equiv (X \equiv (Y \equiv Z))$$

It follows that any expression built up from P and Q using only $\equiv$ can be reparenthesized, and reordered. Just as we do in elementary algebra when dealing with addition, in effect we can leave parentheses out.

So, let E be an expression built up from P and Q using only $\equiv$. E can be put into the following form: $P \equiv P \equiv \ldots \equiv P \equiv Q \equiv Q \equiv \ldots \equiv Q$. Now, $(P \equiv P) \equiv X$ is easily shown to be equivalent to X, and similarly for $(Q \equiv Q) \equiv X$. Consequently we can eliminate P's and Q's in pairs, and conclude that E must be equivalent to one of the following for expressions: *true*, P, Q, or $P \equiv Q$. It follows that E is not equivalent to $P \circ Q$ where $\circ$ is any Primary Connective.

Next, suppose we allow $\neg$ as well as $\equiv$. It is easy to check that the following are all equivalent: $X \equiv \neg Y$, $\neg X \equiv Y$, and $\neg(X \equiv Y)$. It follows that starting with any expression E built up from P and Q using $\neg$ and $\equiv$, we can 'push' negations further and further out, until they are all at the outer level. Then we can eliminate double negations. We conclude that E must be equivalent to either F or $\neg G$ where F and G contain no negations. This reduces things to the case we just considered. E must be equivalent to one of *true*, P, Q, $P \equiv Q$, *false*, $\neg P$, $\neg Q$, $P \not\equiv Q$. And again we do not get any of the Primary Connectives.

Finally, nothing essential changes if we add $\not\equiv$, since it is already definable using $\neg$ and $\equiv$.