# Corrections to
# *Set Theory and the Continuum Problem* (revised edition)

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These are corrections to the edition published by Dover in 2010.

**Page 23** (Error found by David Feuer) Exercise 5.5(d) in Chapter 2 asks the reader to show $B - (A - B) = \emptyset$. It should be to show $B - (A - B) = B$.

**Page 23** (Error found by David Feuer) Text in §6 reads $A_6$ [**Power aet axiom**]. It should read $A_6$ [**Power set axiom**].

**Page 51** (Error found by Grigory Olkhovikov) Exercise 1.2 is incorrect as asked. Take $A$ to be the set of negative integers under the natural linear order $\leq$. Every proper lower section of $A$ has a strict upper bound, but its ordering is not a well ordering. The exercise can be corrected by including the condition that $A$ has a least element

**Page 60** (Error found by Allen David Boozer) Exercise 4.3 should be deleted. There is a reference to this Exercise in the Remark at the top of page 63, and this reference should also be deleted.

**Page 66** (Error found by Stuart Newberger) Definition 7.1 is incorrect as stated. It should read as follows.

For any sets $y$ and $x$, we will say that $y$ is *closed* (under $g$) *relative to* $x$ provided, for any $z \in y$, if $g(z) \in \mathcal{P}(x)$ then $g(z) \in y$. (Thus $(z \in y \wedge g(z) \subseteq x) \supset g(z) \in y$.)

**Page 225-226** (Error found by Chang Soon Choi.) In the Remarks, it is not the case that $(f \approx_\lambda g) \supset [\![f \approx_\lambda g]\!]$ is $\mathsf{S4}$ valid generally, but it is valid in the particular $\mathsf{S4}$ models being constructed. Here is the argument. Suppose as an induction hypothesis that it is known for ordinals less than $\lambda$. It follows from the definition of $\approx_\lambda$ that if $\alpha < \lambda$ then $(f \approx_\alpha g) \supset (f \approx_\lambda g)$ is valid in these models. It follows from this, using general $\mathsf{S4}$ reasoning, that $\Box\Diamond(f \approx_\alpha g) \supset \Box\Diamond(f \approx_\lambda g)$ is also valid in these models, that is, $[\![f \approx_\alpha g]\!] \supset [\![f \approx_\lambda g]\!]$. Now if $p \Vdash (f \approx_\lambda g)$, then $p \Vdash (f \approx_\alpha g)$ for some $\alpha < \lambda$ (by definition). This implies $p \Vdash [\![f \approx_\alpha g]\!]$, and hence $p \Vdash [\![f \approx_\lambda g]\!]$.

**Page 227** (Error found by Chang Soon Choi.) In the proof of Proposition 1.5, the limit ordinal case is incorrect. It uses an inference from $p \Vdash [\![f \approx_\lambda g]\!]$, where $\lambda$ is a limit ordinal, to $p \Vdash [\![f \approx_\alpha g]\!]$, for some $\alpha < \lambda$, and this is not justified. Replace the limit ordinal case by the following.

Assume $\lambda$ is a limit ordinal and every $\alpha < \lambda$ is good. Now suppose $p \Vdash [\![f \approx_\lambda g]\!]$ and $\lambda < \beta$; we must show $p \Vdash [\![f \approx_\beta g]\!]$.

We first show $(f \approx_\lambda g) \supset [\![f \approx_\beta g]\!]$ is valid in the model that has been constructed (note the absence of double square brackets in the antecedent). Well, suppose $q \Vdash (f \approx_\lambda g)$. Then $q \Vdash (f \approx_\alpha g)$ for some $\alpha < \lambda$, by definition of $\approx_\lambda$. It follows by the Remarks at the bottom of page 225 and the top of 226 that $q \Vdash [\![f \approx_\alpha g]\!]$. Since $\alpha$ is good, $q \Vdash [\![f \approx_\beta g]\!]$. Since $q$ was arbitrary, we have shown the validity of $(f \approx_\lambda g) \supset [\![f \approx_\beta g]\!]$ in the model.

It now follows, by standard modal manipulations, that $\Box\Diamond(f \approx_\lambda g) \supset \Box\Diamond[\![f \approx_\beta g]\!]$ is also valid in the model, and hence we have the validity of $[\![f \approx_\lambda g]\!] \supset [\![f \approx_\beta g]\!]$, making use of Proposition 4.3, part 2. Since $p \Vdash [\![f \approx_\lambda g]\!]$, then $p \Vdash [\![f \approx_\beta g]\!]$.

**Page 228** (Found by Grigori Mints) In the Remark just before Definition 1.8 it is asserted that $(f \in g) \equiv [\![f \in g]\!]$. The equivalence is not correct, but $(f \in g) \supset [\![f \in g]\!]$ is.

**Page 229** (Problem found by Chang Soon Choi.) Lemma 2.1 says that if $p \Vdash [\![f \approx_\alpha g]\!]$ and $p \Vdash [\![g \approx_\alpha h]\!]$ then $p \Vdash [\![f \approx_\alpha h]\!]$. The proof is by induction on $\alpha$. It begins by saying the cases where $\alpha$ is 0 or a limit ordinal are simple. In fact 0 is simple, but the limit ordinal case is not. Here is a proof for the limit ordinal case.

Let $\lambda$ be a limit ordinal and assume the result holds for smaller ordinals. We begin by showing that if $p \Vdash (f \approx_\lambda g)$ and $p \Vdash [\![g \approx_\lambda h]\!]$ then $p \Vdash [\![f \approx_\lambda h]\!]$ (note the difference in the first item). So, suppose $p \Vdash (f \approx_\lambda g)$ and $p \Vdash [\![g \approx_\lambda h]\!]$. To show $p \Vdash [\![f \approx_\lambda h]\!]$ we show $p \Vdash \Box\Diamond(f \approx_\lambda h)$. Let $q$ be any member of $\mathcal{G}$ such that $p\mathcal{R}q$; we must show there is some $r$ with $q\mathcal{R}r$ so that $r \Vdash (f \approx_\lambda h)$.

Since $p \Vdash \Box\Diamond(g \approx_\lambda h)$ then $q \Vdash \Diamond(g \approx_\lambda h)$ and hence there is some $r$ with $q\mathcal{R}r$ so that $r \Vdash (g \approx_\lambda h)$. By definition, $r \Vdash (g \approx_\alpha h)$ for some $\alpha < \lambda$. Without loss of generality we can assume $\alpha$ is a successor ordinal. Then $r \Vdash [\![g \approx_\alpha h]\!]$ by the remarks on pages 225-226.

Since $p \Vdash (f \approx_\lambda g)$ then $p \Vdash (f \approx_\beta g)$ for some $\beta < \lambda$ and again without loss of generality we can assume $\beta$ is a successor ordinal. Then $p \Vdash [\![f \approx_\beta g]\!]$ by the remarks on pages 225-226 again. Since this formula begins with $\Box$, $r \Vdash [\![f \approx_\beta g]\!]$. Let $\gamma$ be the larger of $\alpha$ and $\beta$. By Proposition 1.5, $r \Vdash [\![f \approx_\gamma g]\!]$ and $r \Vdash [\![g \approx_\gamma h]\!]$. Since $\gamma < \lambda$, by the induction hypothesis for the overall Lemma, $r \Vdash [\![f \approx_\gamma h]\!]$. Since $\gamma$ is a successor ordinal, by the remarks on pages 225-226 again, $r \Vdash (f \approx_\gamma h)$, which is what we wanted.

Since $p$ was arbitrary, we have shown the validity in our model of

$$(f \approx_\lambda g) \supset \left([\![g \approx_\lambda h]\!] \supset [\![f \approx_\lambda h]\!]\right).$$

Then by standard $\mathsf{S4}$ manipulations, this gives us the validity in our model of

$$\Box\Diamond(f \approx_\lambda g) \supset \Box\Diamond\left([\![g \approx_\lambda h]\!] \supset [\![f \approx_\lambda h]\!]\right).$$

By Proposition 4.4 of Chapter 16 we then have

$$\Box\Diamond(f \approx_\lambda g) \supset [\![g \approx_\lambda h \supset f \approx_\lambda h]\!]$$

and hence

$$[\![f \approx_\lambda g]\!] \supset \left([\![g \approx_\lambda h]\!] \supset [\![f \approx_\lambda h]\!]\right)$$

by using Proposition 4.5 of Chapter 16.

**Page 234** (Problem found by Chang Soon Choi.) In line 8 of the proof of Proposition 3.3, "equivalently, $[\![\hat{s} \in \hat{t}]\!]$" should be changed to "and so $[\![\hat{s} \in \hat{t}]\!]$". Also in line 11 of the same proof, "But then $p \Vdash (a \,\varepsilon\, \hat{t})$, so $a$ is $\hat{x}\ldots$" should be changed to "So $a$ is $\hat{x}\ldots$".

**Pages 239–240** (Problem found by Chang Soon Choi.) In the proof of Lemma 5.2 it is said that "(Recall that $(x \approx_\alpha y)$ and $[\![x \approx_\alpha y]\!]$ are equivalent.)" This is not the case. One should modify the condition that we need to express by a first-order formula so that the last part reads $\Box\Diamond(x \approx_\alpha y)$. Then, in the formula following "Now let $F(\mathcal{A}, p, f, g)$ be the formula:" the final clause should be changed from "$\langle s', x, y\rangle \in \mathcal{A}$" to "$\langle r', x, y\rangle \in \mathcal{A}$".

**Page 241** (Problem found by Chang Soon Choi.) In the proof of Theorem 5.4, the atomic case should be modified. We know that $p \Vdash [\![f \in g]\!]$ iff $p \Vdash [\![(\exists w)(w \approx f \wedge w \,\varepsilon\, g)]\!]$ iff $p \Vdash \Box\Diamond(\exists w)([\![w \approx f]\!] \wedge [\![w \,\varepsilon\, g]\!])$, so in the atomic case, $F_\varphi(z, x, y)$ should be

$$(\forall z' \overleftarrow{\mathcal{R}} z)(\exists z'' \overleftarrow{\mathcal{R}} z')(\exists w \in \mathcal{D})(\mathsf{Equals}(z'', w, x) \wedge (\forall z''' \overleftarrow{\mathcal{R}} z'')(\exists z'''' \overleftarrow{\mathcal{R}} z''')\langle z'''', w\rangle \in y)$$

Also in the final part of the proof take $F_\varphi(z, x_1, \ldots, x_n)$ to be the following: $(\forall z' \overleftarrow{R} z)(\exists z'' \overleftarrow{R} z')\neg F_\psi(z'', x_1, \ldots, x_n)$.

**Page 264** (Problem found by Jason Parker) The remarks at the end of the first paragraph are incorrect. First, a few useful observations: using Definition 3.1 on page 233, one has the following.

$$\begin{aligned}
\hat{0} &= \emptyset \\
\hat{1} &= \mathcal{G} \times \{\hat{0}\} \\
\hat{2} &= \mathcal{G} \times \{\hat{0}, \hat{1}\} \\
&= \hat{1} \cup (\mathcal{G} \times \{\hat{1}\}) \\
\hat{3} &= \mathcal{G} \times \{\hat{0}, \hat{1}, \hat{2}\} \\
&= \hat{2} \cup (\mathcal{G} \times \{\hat{2}\}) \\
&\vdots \\
\hat{\omega} &= \mathcal{G} \times \{\hat{0}, \hat{1}, \hat{2}, \ldots\}
\end{aligned}$$

Now, here is Parker's argument.

"It is claimed that we can show that if $p \Vdash [\![a \subseteq \hat{\omega}]\!]$, then for some $b \subseteq \mathcal{G} \times \hat{\omega}$, $p \Vdash [\![a \approx b]\!]$. But this would entail that $b \in \mathcal{D}^{\mathcal{G}}$, which does not seem possible. For suppose $b \subseteq \mathcal{G} \times \hat{\omega}$. If $b$ were a member of $\mathcal{D}^{\mathcal{G}}$, then $b \in R^{\mathcal{G}}_{\alpha+1}$ for some least ordinal $\alpha$. Then $b \subseteq \mathcal{G} \times R^{\mathcal{G}}_\alpha$. Now since $b \subseteq \mathcal{G} \times \hat{\omega}$, it follows that any $x \in b$ is of the form $\langle p, \langle q, \hat{n}\rangle\rangle$ for some $p, q \in \mathcal{G}$ and $n \in \omega$. So $\langle q, \hat{n}\rangle \in R^{\mathcal{G}}_\alpha$. So there is some least ordinal $\beta$ such that $\langle q, \hat{n}\rangle \in R^{\mathcal{G}}_{\beta+1}$, whereby $\langle q, \hat{n}\rangle \subseteq \mathcal{G} \times R^{\mathcal{G}}_\beta$, which is clearly false. So it seems that it cannot be that $b \in \mathcal{D}^{\mathcal{G}}$ if $b \subseteq \mathcal{G} \times \hat{\omega}$."

The problem sentences at the end of paragraph 1, page 264, should be replaced with the following. "Now, this result can be improved, to establish that if $[\![a \subseteq \hat{\omega}]\!]$ is true at $p$ then $[\![a \approx b]\!]$ is true at $p$ for some $b \subseteq \mathcal{G} \times \{\hat{n} \mid n \in \omega\}$ (equivalently, for some $b \subseteq \hat{\omega}$). Consequently, to investigate the size of the power set of $\hat{\omega}$ in the modal model, we begin by investigating the actual power set of $\hat{\omega}$."

Here is the argument for the revised assertion above. Throughout, assume that $p \Vdash [\![a \subseteq \hat{\omega}]\!]$, meaning $p \Vdash [\![(\forall x)(x \in a \supset x \in \hat{\omega})]\!]$.

1. If $p\mathcal{R}p'$ and $p' \Vdash [\![x \in a]\!]$, then for some $p''$ with $p'\mathcal{R}p''$, $p'' \Vdash [\![x \approx \hat{n} \wedge \hat{n} \in a]\!]$ for some $n \in \omega$.

Proof: Suppose $p\mathcal{R}p'$, and $p' \Vdash [\![x \in a]\!]$. Then $p' \Vdash [\![x \in \hat{\omega}]\!]$, and so for some $p''$ with $p'\mathcal{R}p''$, $p'' \Vdash x \in \hat{\omega}$, and hence for some $h$, $p'' \Vdash [\![x \approx h]\!]$ and $p'' \Vdash [\![h \,\varepsilon\, \hat{\omega}]\!]$ (Definition 1.6 Chapter 17). Then for some $p'''$ with $p''\mathcal{R}p'''$, $p''' \Vdash h \,\varepsilon\, \hat{\omega}$, and so $\langle p''', h\rangle \in \hat{\omega} = \mathcal{G} \times \{\hat{0}, \hat{1}, \ldots\}$. Then $h = \hat{n}$ for some $n \in \omega$. It follows that $p'' \Vdash [\![x \approx \hat{n}]\!]$ and $p'' \Vdash [\![\hat{n} \in a]\!]$.

2. Now let $b = \{\langle q, \hat{n}\rangle \mid n \in \omega, q \Vdash [\![\hat{n} \in a]\!]\}$. Trivially $b \subseteq \mathcal{G} \times \{\hat{0}, \hat{1}, \ldots\} = \hat{\omega}$.
3. $p \Vdash [\![a \subseteq b]\!]$. The proof is by contradiction. Suppose not; then for some $h$ and for some $p'$ with $p\mathcal{R}p'$, $p' \Vdash [\![h \in a]\!]$ and $p' \Vdash [\![\neg(h \in b)]\!]$ ($\mathbf{P}_9$, Page 226). By item 1, for some $p''$ with $p'\mathcal{R}p''$, $p'' \Vdash [\![h \approx \hat{n}]\!]$ and $p'' \Vdash [\![\hat{n} \in a]\!]$ for some $n \in \omega$. Let $q$ be an arbitrary member of $\mathcal{G}$ with $p''\mathcal{R}q$. Then $q \Vdash [\![\hat{n} \in a]\!]$, hence $\langle q, \hat{n}\rangle \in b$, and so $q \Vdash \hat{n} \,\varepsilon\, b$. Since $q$ was arbitrary, $p'' \Vdash \Box(\hat{n} \,\varepsilon\, b)$, and so $p'' \Vdash \Box\Diamond(\hat{n} \,\varepsilon\, b)$, or $p'' \Vdash [\![\hat{n} \,\varepsilon\, b]\!]$. Then $p'' \Vdash [\![\hat{n} \in b]\!]$ (Corollary 2.4, Chapter 17). But we also have $p'' \Vdash [\![\neg(\hat{n} \in b)]\!]$, and this is our contradiction.
4. $p \Vdash [\![b \subseteq a]\!]$. Again the proof is by contradiction. If not, then for some $h$ and for some $p'$ with $p\mathcal{R}p'$, $p' \Vdash [\![h \in b]\!]$ and $p' \Vdash [\![\neg(h \in a)]\!]$. Then for some $p''$ with $p'\mathcal{R}p''$, $p'' \Vdash h \in b$ and hence for some $k$, $p'' \Vdash [\![h \approx k]\!]$ and $p'' \Vdash [\![k \,\varepsilon\, b]\!]$. Then for some $p'''$ with $p''\mathcal{R}p'''$, $p''' \Vdash k \,\varepsilon\, b$. But then $\langle p''', k\rangle \in b$, and so $k = \hat{n}$ for some $n \in \omega$, and $p''' \Vdash [\![\hat{n} \in a]\!]$ (definition of $b$). We also have $p'' \Vdash [\![h \approx \hat{n}]\!]$, and it follows that $p''' \Vdash [\![\neg(\hat{n} \in a)]\!]$, a contradiction.

Additional changes resulting from the correction described above.

Page 264, second paragraph should begin: "Let $C = \{a \mid a \subseteq \mathcal{G} \times \{\hat{0}, \hat{1}, \ldots\}\}$."

Page 264, last paragraph of the Proof of Lemma 5.2, second sentence. This should begin: "Since $a \subseteq \mathcal{G} \times \{\hat{0}, \hat{1}, \ldots\}\ldots$".

Page 265, paragraph following Proposition 5.5. This should read: "We are finished investigating $\mathcal{P}(\mathcal{G} \times \{\hat{0}, \hat{1}, \ldots\}$ and its subset $C_0$.

**Page 272** (Problem found by Grigori Mints). The proof of Proposition 20.4.1 begins by noting that $f \in g$ and $[\![f \in g]\!]$ are equivalent. This is not the case, see correction to Page 228. However, the atomic case is still straightforward. For $f, g \in \mathcal{D}^{G}_{\mathfrak{F}}$, $p \Vdash f \in g$ if and only if $p \Vdash_{\mathfrak{F}} f \in g$ for every $p$, by the definition of $\Vdash_{\mathfrak{F}}$. It follows that $p \Vdash \Box\Diamond(f \in g)$ if and only if $p \Vdash_{\mathfrak{F}} \Box\Diamond(f \in g)$ for every $p$.

The following errors and typos were reported by Jonathan Farley.

**Page 23, line 17** "aet" should be "set"

**Page 25, line 14** "A2" should be "A5"

**Page 30, line 19** "qualifiy" should be "qualify"

**Page 38, line 2** It is true as written, but "$y \subset x$" should be "$y \subseteq x$"

**Page 40, line 11** "bounded" should be "non-empty bounded"

**Page 44, line 15** Do we know $c$ is a set? Response: standard mathematical practice treats this as a set, but technically it is not justified until the Axiom of Substitution is introduced, **Ax 8** on page 170.

**Page 49, line 14** "$x'Rb'$" should be "$b'Rx'$"

**Page 49, line 15** "$x \leq b$" should be "$b \leq x$"

**Page 58, line -2** "$M$" should be "$N$"

**Page 59, line -5** "$M$" should be "$S$"

**Page 60, line -8** "$N$ should be "$\cup N$"

**Page 60, line -5** (Further corrected by Rolf Rolles) This should read "Every successor element of $N$ is $F(a)$ for some $a \in \cup N$."

**Page 60, Exercise 4.3** Delete this exercise. Here is a counter-example. Take any set $x$, and consider $S = \{x\}$. The axiom of choice is not needed to say $S$ has a choice function, but $\cup S = x$, and this need not have a choice function

**Page 62, line 9** The semicolon should be a comma

**Page 62, Lemma 5.4** Add the assumption that $A \neq \emptyset$

**Page 62, line -2** "5.3" should be "5.2"

**Page 63, line 5** Exercise 4.3 has been deleted

**Page 63, line 7** After "denumerable" add "or finite"

**Page 64, line -9** Before "set of finite character" insert "non-empty"

**Page 66** It would have helped to point out at the beginning of §7 that $g$ is defined on all sets

**Page 73, line 3** "10.2" should be "10.3"

**Page 79, following Definition 1.1** Should begin, "In general, $F(x) \neq F''(x)$"

**Page 79, Second paragraph following Definition 1.1** Should contain "whereas $F''(x)$ is"

**Page 80, line 2** Add that $\varphi_2$ is $1-1$

**Page 80, proof of Proposition 1.3, second line** "$L$" should be "$L_<$"

**Page 80, line -2** "onto" should be "into"

**Page 88, line -7** "isomporphic" should be "isomorphic"

**Page 89, Theorem 6.1** This should begin "For any functions $f(x), g(x)$ on ordinals, and any function $h(x, y, z)$, where $y$ and $z$ are ordinals, ...

**Page 93, $\mathbf{O}_6$** "Since $x$" should be "Since $S$"

**Page 97, last line of Proof of $\mathbf{Q}_4$** "has rank $< \alpha$" should be "has rank $\leq \alpha$"

**Page 97, line -8** "$mathbbF$" should be "$\mathbb{F}$"

**Page 99, next to last line of Example** "every subclass" should be "every non-empty subclass"

**Page 99, line -14** "Zermelo Fraenkel" needs a hyphen

**Page 101, line -16** "$x$ of $A$" should be "$x$ of $A - B$"

**Page 102, line -6** "set" should be "class"

**Page 306, Tarski, A. (1955)** "lattice-theoretical theorem" should be "lattice-theoretical fixpoint theorem"