Notes on ‘Set Theory & Continuum Problem’

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1 Errata or what looks like these

p. 137. Theorem 6.1. of Chapter 10 seems to be incorrect as stated. One counterexample is: let $A = B = 2$ and $R = \in$, $G = \ni$. Then isomorphism F from (A, R) onto (B, G) looks like this: $F(0) = 1$, $F(1) = 0$. Also, (A, R) clearly satisfies assumptions of the theorem and B, being an ordinal, is transitive. But F is not the Mostowski-Shepherdson map for (A, R), since

$$0 = F(1) \neq F''(1^*) = F''(1) = F''(\{0\}) = \{1\}.$$

The theorem can be amended by putting $(B, \in)$ instead of (B, G). (In fact, this amendment seems to be assumed in the proof given for the theorem, so this might be classified as a misprint as well).

p.174–175. Lemma 3.2, Chapter 13, seems to use transitivity of K after all. When replacing formulas of the form $a \in x_i$ with formulas $(\exists y \in x_i)(y = a)$ it will not do to unfold the formula as $(\exists y \in x_i)\forall z(y \in z \equiv a \in z)$, for such a formula will be irrelevant (since it still contains $a \in z$). One should rather understand the $=$ in the formula along the lines of $(\exists y \in x_i)\forall z(z \in y \equiv z \in a)$, but the fact that $\forall z(z \in y \equiv z \in a)$ defines $y = a$ over K presupposes transitivity (or at least extensionality) of K. So, one can either mention transitivity of K as an assumption of the lemma, or, alternatively, one can include $K^n_{\bar{a},j}$ into the types of distinguished subclasses mentioned in the hypothesis (1) of Theorem 3.1. This will not get into the way of the application of this theorem to L due to the presence of Exercise 3.1 of Chapter 12.

p. 185. The statement: ‘The four conditions of (0) can be collectively stated: $t(m_1 \in m_2) = \{\ulcorner m_1 \urcorner, \ulcorner m_2 \urcorner\} \cap \omega$’ is incorrect, for, e. g. $t(x_1 \in x_2) = \{1, 2\}$, but $\{\ulcorner x_1 \urcorner, \ulcorner x_2 \urcorner\} \cap \omega = \{\langle 0, 1\rangle, \langle 0, 2\rangle\} \cap \omega = \emptyset$, since no natural number is an ordered pair of natural numbers. Therefore, Σ-condition (2) in the proof of Lemma 3.4 on this page must be replaced, e. g. by the following formula:

$$\begin{aligned}
&\exists z(\forall x \in c(a))(\forall y \in c(a))(f(\langle x, y, 0\rangle) = z \wedge \\
&\quad \wedge (\forall w \in z)\exists x'(x' = 0 \wedge (x = \langle x', w\rangle \vee y = \langle x', w\rangle)) \\
&\quad \wedge \exists y_1 y_2 y_3(y_1 = \{x, y\} \wedge y_2 = \cup y_1 \wedge y_3 = \cup y_2 \\
&\qquad \wedge (\forall y_4 \in y_3)((\exists y_5 \in y_3)(y_5 = 0 \wedge (x = \langle y_5, y_4\rangle \vee y = \langle y_5, y_4\rangle)) \supset y_4 \in z) \\
&\quad) \\
&)
\end{aligned}$$

(I was aiming at correctness rather than brevity). This formula is Σ due to the items (7), (9), (14), (24) from p. 159.

p. 275. The part of proof of Lemma 5.1, Chapter 20, given on this page employs the following transition[1]: from

$$q \Vdash [[c \approx x' \wedge c\varepsilon a]]$$

conclude to

$$\theta^{-1}(q) \Vdash [[\theta^{-1}(c) \approx \theta^{-1}(x') \wedge \theta^{-1}(c)\varepsilon a]],$$

using Theorem 3.2 and $\mathfrak{C}$-invariance of a. Now, this theorem warrants transition to

$$\theta^{-1}(q) \Vdash [[\theta^{-1}(c) \approx \theta^{-1}(x') \wedge \theta^{-1}(c)\varepsilon\theta^{-1}(a)]],$$

and $\mathfrak{C}$-invariance of a means that $[[a \approx \theta^{-1}(a)]]$ is S4-valid in $\mathcal{M}$, but this does not give us our conclusion, since the book proves the substitutivity of $\approx$ only with respect to formulas of classical language, which do not contain ε. This shows that the transition is unwarranted but not necessarily shows that it is wrong. I think that I can produce both a counterexample to the transition and a possible amendment to the proof.

Counterexample. Let $\mathcal{G} = \{p, q, r, s\}$, let $\mathcal{R}$ be a transitive and reflexive closure of $\{\langle p, q\rangle, \langle r, s\rangle\}$ and let $\theta(p) = r$, $\theta(r) = p$, $\theta(q) = s$, and $\theta(s) = q$. Then θ is an automorphism and the group generated by θ contains only θ itself plus an equivalence function as an identity map. Also, θ happens to be its own inverse. Consider then the following sets:

$$a = \{\langle p, \hat{p}\rangle, \langle q, \hat{p}\rangle, \langle s, \hat{p}\rangle\}$$
$$b = \{\langle r, \hat{p}\rangle, \langle q, \hat{p}\rangle, \langle s, \hat{p}\rangle\}$$

It follows from Proposition 2.9, Chapter 20, that $\theta(a) = b$. It is also clear that both $[[\hat{p}\varepsilon a]]$ and $[[\hat{p}\varepsilon b]]$ are S4-valid in $\mathcal{M}$. One can also see that $[[a \approx b]]$ is is S4-valid in $\mathcal{M}$. Indeed, use Proposition 2.12, Chapter 17, and assume that for some $p' \in \mathcal{G}$ we have $p' \Vdash [[x\varepsilon a]]$. Then, of course, $x = \hat{p}$ and we have both $p' \Vdash [[\hat{p} \approx \hat{p}]]$ and $p' \Vdash [[\hat{p}\varepsilon b]]$, and similarly for the other direction. So we have shown that a is invariant with respect to the group of automorphisms generated by θ (since $\hat{p}$, the only ε-element of a, is of course invariant by Proposition 2.9). Now consider the two sets

$$c = \mathcal{G} \times \{a\}$$
$$d = \mathcal{G} \times \{b\}$$

Again we have $\theta(c) = d$ and due to the S4-validity of $[[a \approx b]]$ we clearly have both the S4-validity of $[[c \approx d]]$ and the invariance of c with respect to our group of automorphisms. So, to summarize, we have, e g.

$$p \Vdash [[a \approx a \wedge a\varepsilon c]],$$

but we do not have

$$r \Vdash [[b \approx b \wedge b\varepsilon c]],$$

since $b\varepsilon c$ never holds in $\mathcal{M}$. And given that $\theta(a) = b$, $\theta(p) = r$ and $\theta = \theta^{-1}$, this means that

$$\theta^{-1}(p) \nVdash [[\theta^{-1}(a) \approx \theta^{-1}(a) \wedge \theta^{-1}(a)\varepsilon c]].$$

[1] I am using double square brackets instead of the involved brackets the authors use to denote the translation from modal to non-modal formulas

Amendment. We revise the lemma by adding the assumption that automorphisms in $\mathfrak{F}$ are first-order definable in M and we begin the amended proof by establishing the following:

Claim. If $a \in R^{\mathcal{G}}_{\alpha}$, and $\theta \in \mathfrak{C}_0$, then $\theta(a) \in R^{\mathcal{G}}_{\alpha}$.

Proof. By induction on α. Induction basis and the limit case are obvious; we consider successor case. Let $\alpha = \beta + 1$ and let $a \in R^{\mathcal{G}}_{\beta+1}$. If $b \in \theta(a)$, then $b = \langle \theta(p), \theta(c) \rangle$, where $p \in \mathcal{G}$ and $c \in R^{\mathcal{G}}_{\beta}$. By induction hypothesis, $\theta(c) \in R^{\mathcal{G}}_{\beta}$. Therefore,

$$\theta(a) \subseteq \mathcal{G} \times R^{\mathcal{G}}_{\beta}.$$

To arrive at our conclusion, $\theta(a) \in R^{\mathcal{G}}_{\beta+1}$, it remains to show that $\theta(a) \in M$; given that θ is first-order definable, $a \in M$ and M is a first-order universe, this requires only an application of the corresponding version of substitution schema. □

Having the Claim established, we proceed more or less in the same way as in the book, until we get the following premises:

$$q \Vdash x' \varepsilon b \tag{1}$$
$$q \Vdash [[x \in b]] \tag{2}$$
$$q \Vdash [[\neg x \in \theta(b)]] \tag{3}$$
$$q \Vdash [[x \approx x']] \tag{4}$$
$$q \Vdash [[x' \varepsilon b]] \tag{5}$$
$$q \Vdash [[c \approx x' \wedge c \varepsilon a]] \tag{6}$$
$$x' \in R^{\mathcal{G}}_{\alpha} \cap \mathcal{D}^{\mathcal{G}}_{\mathfrak{F}} \tag{7}$$
$$\mathcal{M} \models_{S4} [[a \approx \vartheta(a)]] \qquad \text{(for every } \vartheta \in \mathfrak{C}) \tag{8}$$

Now we reason as follows:

$$q \Vdash [[x' \in a]] \qquad \text{(from (6))} \tag{9}$$
$$\theta^{-1}(q) \Vdash [[\theta^{-1}(x') \in \theta^{-1}(a)]] \qquad \text{(from (9) by T3.2)} \tag{10}$$
$$\theta^{-1}(q) \Vdash [[\theta^{-1}(x') \in a]] \qquad \text{(from (8), (10) by T2.7 Ch. 17)} \tag{11}$$

Now, using (11), take some $d \in \mathcal{D}^{\mathcal{G}}$ and $r \in \mathcal{G}$ such that

$$\theta^{-1}(q) \mathcal{R} r \tag{12}$$
$$r \Vdash [[\theta^{-1}(x') \approx d \wedge d \varepsilon a]] \tag{13}$$

We then continue in the following way:

$$\theta^{-1}(x') \in R^{\mathcal{G}}_{\alpha} \cap \mathcal{D}^{\mathcal{G}}_{\mathfrak{F}} \quad \text{(from (7) by the above Claim \& T3.3)} \qquad (14)$$

$$r \Vdash [[\theta^{-1}(x')\varepsilon b]] \quad \text{(from (13) and (14) by df of } b) \qquad (15)$$

$$r \Vdash [[\theta^{-1}(x') \in b]] \quad \text{(from (15))} \qquad (16)$$

$$\theta(r) \Vdash [[x' \in \theta(b)]] \quad \text{(from (15) by T3.2)} \qquad (17)$$

$$q\mathcal{R}\theta(r) \quad \text{(from (12))} \qquad (18)$$

$$\theta(r) \Vdash [[x \in \theta(b)]] \quad \text{(from (4), (17), and (18))} \qquad (19)$$

$$\theta(r) \Vdash [[\neg x \in \theta(b)]] \quad \text{(from (3) and (18))} \qquad (20)$$

So we got our contradiction in place.

A final erratum which I would like to call attention to is not a very harmful (in the sense that it does not render the results wrong) but still somewhat misleading and very pervasive habit of the authors to skip a transition to an $\mathcal{R}$-successor demanded by the contexts of the form $p \Vdash [[a \in b]]$. E. g. on p. 230 authors write: 'Now, $p \Vdash [[f \in h]]$, so $p' \Vdash [[f \in h]]$. Then, for some a, $p' \Vdash [[a \approx f]]$ and $p' \Vdash [[a\varepsilon h]]$'. In fact, as the authors note themselves on p. 228, it is $f \in h$ that is equivalent to $\exists a[[a \approx f \wedge a\varepsilon h]]$, whereas $[[f \in h]]$ is equivalent to $[[\exists a(a \approx f \wedge a\varepsilon h)]]$, and so P_{11} on p. 226 warrants only the following conclusion: Then, for some a and some p'' such that $p'\mathcal{R}p''$:

$$p'' \Vdash [[a \approx f]] \wedge [[a\varepsilon h]].$$

This skipping of successor transitions, however, does not get into the way of conclusions the authors aim at, since conditions of the form $[[X]]$ are monotonic with respect to accessibility relation. Still, it is worth noting that this skipping occurs (as far as I could notice) on the following pages of the book: 230, 232, 233, 234, 236, 237, 239, 243, 246, 247, 257, 275.

2 Misprints

p. 133 In the proof of Theorem 3.1, Chapter 10 while constructing an On sequence, authors demand that

$$f_{\alpha+1} = f_{\alpha},$$

while the subsequent reasoning seems to suggest rather

$$f_{\alpha+1} = f_{\alpha} \cup g(f_{\alpha}),$$

where $g(f_{\alpha}) = \{\langle x, g(f''_{\alpha}(x^*))\rangle \mid x \in \Gamma_{\alpha+1} \setminus \Gamma_{\alpha}\}$.

p.138. In the proof of Theorem 6.3, Chapter 10, the phrase '(by P_2 of §2)' needs to be replaced with '(by P_8 of §2)'.

p.152 In the proof of Theorem 5.1, Chapter 11, the two occurrences of W in the phrase '(because any element... hence of W)' need to be replaced with w_{β}.

p.170, footnote 1 $x = y$ is said to be abbreviation of $\forall z(z \in x \equiv z \in y)$ whereas subsequent reasoning suggests rather $\forall z(x \in z \equiv y \in z)$.

p. 220. In the axiom (5) the last parenthesis is missing.

p. 235. It is necessary to remove hats from $y_1 \dots y_k$ in the proof of Corollary 3.4, Chapter 17.

p. 277. $\theta \in \mathcal{G}$ needs to be replaced with $\theta \in \mathfrak{C}$.

p. 296. $a_G \in b_B$ needs to be replaced with $a_G \in b_G$.

3 Other notes

Exercise 2.1, Chapter 20, might actually require definability of the automorphisms in question. It is hard to supply a clear counterexample, but at least I could not see how to do without it when proving that the extension of θ on $\mathcal{D}^{\mathcal{G}}$ is onto. I was proving the following statement

$$f \in \mathcal{D}^{\mathcal{G}} \Rightarrow \exists g \in \mathcal{D}^{\mathcal{G}}(f = \theta(g))$$

by an induction on α such that $f \in R^{\mathcal{G}}_{\alpha}$. Then in the successor case the induction hypothesis yields that for every h such that $\langle p, h\rangle \in f$ there is an $h' \in \mathcal{D}^{\mathcal{G}}$ such that $h = \theta(h')$. Since we know that f is an M-set, then the class of right projections of elements of f must be an M-set as well; and now, if we know that θ is definable, we can apply first-order substitution schema to get that the class of their θ-images is an M-set, too. Then it easily follows that the set

$$\{\langle \theta(p), \theta(h)\rangle \mid \langle p, h\rangle \in f\}$$

is an M-set which is a subset of $\mathcal{D}^{\mathcal{G}}$, and so it is in $\mathcal{D}^{\mathcal{G}}$ itself.

Perhaps you had in mind some other solution to the exercise which does not require definability; but if you thought of the solution along the lines outlined above, then it might be a good idea to include the definability assumption into the formulation of the exercise.