List of errors in *Set Theory and the Continuum Problem* Raymond M. Smullyan and Melvin Fitting. November 12, 2005

In the items below, errors were identified by: [VG] = Victoria Gittman; [PG] = Peter Gregory; [EA] = Evangelia Antonakos.

Page 40 In line 7 "(since $f(c) = 0$)" should be "(since $f(0) = c$)". [EA]

Page 40 Exercise 8.3 requires universally quantifying over classes if A is a class. This is not available in the system as formulated. The approach of the exercise works if A is a set. [VG]

Page 85 Theorem 4.1 part 2, $\lambda > \omega \cdot 2$ should be $\lambda \geq \omega \cdot 2$. [PG]

Page 86 Lemma 4.4, and hence Theorem 4.3, are wrong. The problem comes in showing R_ω is in every Zermelo universe—the axiom of substitution seems to be necessary to establish such a result, and it is precisely what we do not have. [VG]

Page 91 Theorem 3.6 is probably incorrect as stated, but becomes correct if "Zermelo universe" is changed to "Zermelo-Fraenkel universe." If V_0 is a Zermelo universe and all ordinals up to α are in V_0, it does not (seem to) follow that $R_\alpha \subseteq V_0$. This certainly follows, however, if the axiom of substitution is available. [VG]

Page 91 Theorem 3.6. "We let Ω be the set α of all ordinals of our larger universe" should read "We let Ω be the class of all ordinals of our larger unverse."

Page 117 Discussion at the bottom of the page concerning the function $p(x)$. The last paragraph should be replaced with the following.

Let us note that for a relational system $(A, \in)$ where A is transitive, $p(x)$ is $\mathcal{P}(x) \cap A$ ($\mathcal{P}(x)$ is the power set of x). Of course for the relational system $(V, \in)$, $p(x) = \mathcal{P}(x)$. [VG]

Page 120 Discussion of **A-Ranks**. It is asserted: "it is automatic that for every subset x of A, $p(x)$ is a set." This is correct if A is transitive, but is incorrect as stated. As a counter-example take A to be the class of all ordered pairs $\langle \alpha, 4 \rangle$ where α is an ordinal. Then $(A, \in)$

is a relational system in which the $\in$ relation never holds (hence it is automatically well founded). $\emptyset \subseteq A$, but $p(\emptyset) = A$ since for every $a \in A$, $a^* = a \cap A = \emptyset$, and so $p(\emptyset)$ is not a set. [VG]

Page 129 Under Abbreviations, item (1) should read: "We use $(\varphi \vee \psi)$ as an abbreviation of $\neg(\neg\varphi \wedge \neg\psi)$."

Page 163 Proposition 1.3: Part (1) is correct. The first half of part (2) is correct. The second half of part (2), and part (3) are incorrect. This induces a number of subsequent changes, affecting pages 164, 165, 166, and 171. These changes follow.

Page 164 Drop Proposition 1.4 and Corollaries 1.5 and 1.6. Replace them with the following.

Definition 1.4 Let us say a formula $\varphi(x, y)$ is *function-like* over a transitive class K if $(\forall x)(\exists y)(\forall z)[\varphi(x, z) \equiv (z = y)]$ is true over K. Equivalently we could use the formula $(\forall x)(\exists y)[\varphi(x, y) \wedge (\forall z)(\varphi(x, z) \supset (z = y))]$.

Theorem 1.5 Suppose $\varphi(x, y)$ is a Σ formula that is function-like over both the transitive class K and over V. Then $\varphi(x, y)$ is absolute over K.

Proof Let $a, b \in K$. If $\varphi(a, b)$ is true over K, since φ is Σ, $\varphi(a, b)$ is true over V because Σ formulas are absolute upwards. Now suppose $\varphi(a, b)$ is true over V. Since $a \in K$ and φ is function-like over K, there must be some $c \in K$ such that $\varphi(a, c)$ is true over K. Since φ is absolute upward, $\varphi(a, c)$ is true over V, but so is $\varphi(a, b)$, so since φ is function-like over V, $b = c$ is true over V. Thus b and c are the same set, hence $\varphi(a, b)$ is true over K.

Page 165 In Theorem 2.8, drop "and hence absolute" from the conclusion.

Page 166 Drop Exercise 2.1.

Page 171 Replace the beginning of the section with the following.

By Theorem 3.12 and Theorem 2.8 it follows that the function $\mathcal{F}^*$—that assigns to each ordinal α the set L_α—is Σ. Suppose we extend this artificially to a function $\mathcal{A}$, on all sets, by mapping non-ordinals to 0.

We let $\mathcal{M}(x, y)$ be a Σ formula, fixed for the discussion, that defines the relation $L_x = y$; that is, $\mathcal{M}(x, y)$ defines the graph of the function $\mathcal{F}^*$.

Also let $\mathcal{N}(x, y)$ be the formula $[\mathsf{Ordinal}(x) \wedge \mathcal{M}(x, y)] \vee [\neg\mathsf{Ordinal}(x) \wedge y = 0]$. Then $\mathcal{N}(x, y)$ defines the function $\mathcal{A}$. We have given an informal proof of the fact that $\mathcal{N}(x, y)$ defines a function, but of course it could be formalized using the axioms of set theory. By Theorem 3.4 of Chapter 13, L is a well founded first-order universe, so it follows that $\mathcal{N}(x, y)$ is function-like over L. Of course $\mathcal{N}(x, y)$ is also function-like over V. Then by Theorem 1.5 (see corrections for page 164 above) $\mathcal{N}(x, y)$ is absolute. It follows that $\mathcal{M}(x, y)$ is also absolute, because it is equivalent to $\mathsf{Ordinal}(x) \wedge \mathcal{N}(x, y)$. We thus have established the following.

Theorem 4.1 The formula $\mathcal{M}(x, y)$ is Σ and absolute over L.

Now continue as in the book.

Page 194 Exercise 2.5 part (d) should read: Show that $(\Diamond X \wedge \Diamond Y) \supset \Diamond(X \wedge Y)$ is not valid, and $\Box(X \vee Y) \supset (\Box X \vee \Box Y)$ is not valid. [VG]

Page 204 Remarks (4) should read: If $f \subseteq \mathcal{G} \times \mathcal{D}^{\mathcal{G}}$ and f is an M-set then $f \in \mathcal{D}^{\mathcal{G}} \ldots$ [VG]

Page 205 In **Remark**, the sentence "And it is easy to see that this extends to $(f \approx_\lambda g)$ where λ is a limit ordinal," should be replaced with "And it is easy to see that for a limit ordinal λ we have that $(f \approx_\lambda g) \supset [\![f \approx_\lambda g]\!]$ is valid. [VG]

Page 233 The first paragraph of §6 should read as follows.

We have shown the axiom of constructibility is independent of *ZF*, even allowing the axiom of well foundedness and the axiom of choice. Now we will show the axiom of constructibility is independent of *ZF* together with the generalized continuum hypothesis as well. Given what has been done in this chapter thus far, all that is left is to show the translate of the generalized continuum hypothesis is valid in the S4-regular model constructed in §4.

The next list of corrections is due to Sakama Tsuyoshi.

Page 55 Exercise 4.1 "$\cup N$ decribed in the proof of Theorem 4.3" should be "Theorem 4.4".

Page 75 line 1 (1) "the least element of M is 0;" For 0, it should be $\emptyset$. (In fact, 0 is $\emptyset$, but perhaps this change makes things clearer.)

Page 77 Note "the On-sequence defined from h and c" should be "defined from g and c".

Page 81 Remarks line 3 "R_{n+1} has 2^n elements." This should be "R_{n+1} has $2^{card(R_n)}$ elements."

Page 81 Proposition 1.2 "For each α, $R_\alpha \subseteq R_{\alpha+1}$" Should read: "For each α, $R_\alpha \in R_{\alpha+1}$ and $R_\alpha \subseteq R_{\alpha+1}$". The proof is as given, plus an appeal too transitivity.

Page 83 $\mathbf{O}_5$ (1) "(by $\mathbf{O}_5$)" should be "(by $\mathbf{O}_4$)".

Page 85 Proof of $\mathbf{Q}_2$ (3) "hence $\cup x \subseteq \cup R_{\beta 1}$," should be "$\cup R_{\beta+1}$".

Page 103 Proof of Theorem 6.2 "hence $A_1 \cup A_2 \cong A_1$" should be "hence $A_1 \cup A_2 \cong A$".

Page 107 "A proper well-ordering of On^2 (2) $\max(\alpha_1, \beta_2) = \max(\alpha_2, \beta_2)$" should be "$\max(\alpha_1, \beta_1) = \max(\alpha_2, \beta_2)$". Similarly, (3) should be "$\max(\alpha_1, \beta_1) = \max(\alpha_2, \beta_2)$".

Page 110 Prop. 9.2 Proof line 7 "x_1 is disjoint from y_2.) Similarly ..." This right paranthesis should be placed after "Similarly $y_1 = y_2$".

Page 111 Prop.9.8 Proof To use Prop.9.1 (2) in this proof, this proof should be supplemented by considering the case that x is emptyset.

Page 117 line 2 "by Propositions 1.1 and 1.2" should be "by Propositions 1.1 and 1.3".

Page 120 Theorem 2.7 Proof line 6 "By P_2" should be "By P_8".

Page 120 line 2 from the bottom "$F(\alpha) = g(F''(\alpha))$" should be "$F(\alpha) = h(F''(\alpha))$".

Page 123 Theorem 5.1 It will be better to note that this A is relation system $(A, \in)$.

Page 126 Corollay 6.4 Proof "such that $F(\alpha) \leq \alpha$," for $\leq$, there should be just $<$.

Page 130 Truth In relational systems (1) "if and only if Ra, b holds." For Ra, b there should be aRb.

Page 135, 4 lines after Theorem 3.2 "only formulas that are of the form $(\exists x)\psi(x, a_1, \ldots, z_k)$," For z_k, there should be a_k.

Page 139 line 38 "w_{β_n} takes care of ψ and $\cup\Gamma''(\beta_n) \subseteq w_{\beta_{n+1}}$." For $\cup\Gamma''(\beta_n)$, there should be $\cup\Gamma''(w_{\beta_n})$.

Page 146 (9) "...$(\forall w \in z)(w = x \vee x = y)$". For "$w = x \vee x = y$", there should be "$w = x \vee w = y$".

Page 147 (19) One more conjunctive should be added: "$\wedge\neg(x = 0)$" (since 0 is not a limit ordinal.)

Page 147 (21) "$x = 0$ iff $\neg(\exists y)(y \in x)$". This is not a Δ_0 formula. It should be "$\neg(\exists y \in x)(y = y)$" (as (7)).

Page 147 (29) The last consequent "$x_1 = x_2$" should be "$x_2 = x_3$".

Page 147 (30) "$y \in Dom(x)$ iff $\ldots(x =< y, v >)$". For "$x =< y, v >$", this should be "$z =< y, v >$".

Page 148 (34)(a) "$Fun(x)$" should be "$Fun(f)$".

Page 151 Proposition 3.8 Proof (1) "$x \in c \wedge (\exists x_1)(\exists x_2)\ldots(\exists x_n)$" should be "$x \in c \wedge (\exists x_1 \in c)(\exists x_2 \in c)\ldots(\exists x_n \in c)$" to be a Δ_0 formula. Similarily for Proof (2).

Page 154 Ax 4 "$\forall x_1 \forall x_2$" should be added to the sentence.

Page 156 F_6 The formula should be "$(\forall y \in x)(y \subseteq a)$".

Page 156 F_8 The formula "$(\exists y)(y \in a \wedge \phi(y, x))$" should be written "$(\exists y \in a)(\phi(y.x))$".

Page 156 F_8 **line 9** The phrase ",and for such an element c, the sentence $c \in a \wedge \phi(a, b)$ is true over K" should be deleted.

Page 158 line 15 The formula "$(\exists y)(y = a \wedge x_i \in y)$" should be "$(\exists y \in x_i)(y = a)$".

Page 166 line 14 The last sentence of the Proof of Theorem 2.8 should also be deleted.

Page 168 line 20 "by $E_b(a)$ we shall mean the set of all a-formulas…". This should be "by $E_b(a)$ we shall mean the set of codes of all a-formulas…".

Page 168 line 21 "We let $\overline{E}_b(a)$ be the set of all a formulas…". This should be "We let $\overline{E}_b(a)$ be the set of codes of all a-formulas…".

Page 169 line 16 "$Sub_a(x, i, b)$ is the code of the formula $\mathrm{Sub}(\phi, v_i, b)$," For "$\mathrm{Sub}(\phi, v_i, b)$", this "Sub" should be italic (since this is a formulua, not its code).

Page 169 Prop 3.7 Proof line 2 The last conjunct in the Sigma-condition should be "$[(\langle 0, i\rangle = x \wedge z = \langle 1, y\rangle) \vee (\langle 0, i\rangle \neq x \wedge z = x)]$".

Page 169 Prop 3.7 Proof line 5 "if $Fun(f) \wedge Dom(f) = E^a$". For "$E^a$", there should be "$(E^a \times \omega) \times a$".

Page 169 Prop 3.7 Proof line 15 "Then $Sub_a(x, i, j) = z$ iff the following holds:". For "$Sub_a(x, i, j) = z$", there should be "$Sub_a(x, i, y) = z$".

Page 170 Valuations and Truth C_0, C_1 The right parenthesis for value of V_a's are missing. (In sum, three parentheses are missing.)

Page 171 Corollary 3.13 Delete this and its Proof.

Page 172 Theorem 4.3 Proof lines 6,8 "$\mathcal{M}(c, b)$" should be "$\mathcal{M}(b, c)$".

Page 172 Theorem 4.3 Proof line 6 "(since $c = M_b$)" should be "(since $c = L_b$)".

Page 172 Theorem 4.3 Proof line 7 "the formula $\mathcal{M}(x, y)$ is absolute,". Not "absolute", but "absolute over L".

Page 172 Theorem 4.3 Proof line 8 "$(\exists y)(\mathcal{M}(y, b) \wedge a \in y)$". For "$\mathcal{M}(y, b)$" should be "$\mathcal{M}(b, y)$".

Page 172f. Theorem 5.1 Proof "$(\forall y)(Ordinal(y) \supset (\exists x)\mathcal{M}(x, y))$". For "$(\exists x)\mathcal{M}(x, y)$" there should be "$(\exists z)\mathcal{M}(y, z)$". (There are three.)

Page 173 Theorem 5.1 Proof "$Ordinal(\alpha) \supset (\exists x)\mathcal{M}(x, \alpha)$". For "$(\exists x)\mathcal{M}(x, \alpha)$", should be "$(\exists z)\mathcal{M}(\alpha, z)$". (There are two.)

Page 173 Theorem 5.1 Proof line 8 (from page top) "$\mathcal{M}(b, \alpha)$" should be "$\mathcal{M}(\alpha, b)$".

Page 173 Theorem 5.1 Proof line 8 "$\mathcal{M}(x, y)$" should be "$\mathcal{M}(y, z)$".

Page 173 line 14, 21 "$\mathbf{C}_{13}$, Chapter 12" should be "$\mathbf{C}_{14}$, Chapter 12".

Page 174 line 32.(5th line in Step 3(1)) "$W \subseteq W_1$" should be "$W \subseteq W_B$".

Page 179 line 11, C2 For "all elements $\alpha \cap K$", this should be "all elements $a \cap K$".

Page 206 line 24 "$p' \Vdash (\forall y)[[(y \epsilon g) \supset \neg(y \approx \beta\alpha)]]$". For "$(y \approx \beta\alpha)$" should be "$(y \approx_\beta \alpha)$".

Page 211 Proposition 2.12 Proof line 6 "being true at p.", For "p" should be "p'".

Page 212 Definition 3.1 "$\hat{x} = < p, \hat{y} > | y \in x$" should be "$\hat{x} = < p, \hat{y} > | p \in G \wedge y \in x$".

Page 221 Proposion 6.1 Proof line 22 (2 lines before end) "so let g be the subset of h such that $v \in h \ldots$". For "$v \in h$" should be "$v \in g$".

Page 255 line 12 "Next, for each $n, k \in \omega$, if $n \neq k$ then $[[\neg(a_n \approx a_k)]]$ is valid in $\mathcal{M}$." For "$(a_n \approx a_k)$" should be "$(s_n \approx s_k)$".

Page 256 line 15 "and so $(\theta^{-1}\pi\theta(m) = \pi(m)$". The right parenthesis for "$\theta^{-1}\pi\theta(m)$" is missing.

Page 262 Definition 3.2 The condition that $\mathcal{G}$ is not empty should be added.

Page 267 line 2 "(3) $p \Vdash (a \epsilon f^*)$". For "$f^*$", should be simply "$f$".

The next corrections are due to Elliott Belbin.

Page 146 (11) This should read: $y = x^+$ iff $x \in y \wedge x \subseteq y \wedge (\forall z \in y)(z \in x \vee z = x)$.

Page 215 Theorem 4.3 The displayed formula is missing a right parenthesis.

Page 217 Exercise 4.2 The displayed formula is missing a right parenthesis.