Proving Axiomatic Unprovability

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1 Axiomatic Provability and Unprovability

Consider the following propositional axiom system.

Axiom Schemes

1. $P \supset (Q \supset P)$
2. $(P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R))$

Rule of Inference $\dfrac{X \quad X \supset Y}{Y}$

In this system $P \supset P$ is provable, but $((P \supset Q) \supset P) \supset P$ (Pierce's law) is not. How could we show this?

To show provability, give a proof.

1. $(P \supset ((P \supset P) \supset P) \supset ((P \supset (P \supset P)) \supset (P \supset P))$ [$Q = (P \supset P)$, $R = P$ in scheme 2]
2. $P \supset ((P \supset P) \supset P)$ [$Q = P \supset P$ in scheme 1]
3. $(P \supset (P \supset P)) \supset (P \supset P)$ [modus ponens on 1, 2]
4. $P \supset (P \supset P)$ [$Q = P$ in scheme 2]
5. $P \supset P$ [modus ponens on 3, 4]

How do we show Pierce's law is *not* provable? Find some property every theorem has, but Pierce's law does not. Well, here's a non-standard truth table for implication, using four truth values. Two are the familiar truth, 1 and falsity, 0, and two are 'intermediate' values.

$\supset$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	b	1

Using this table, we show every axiom evaluates to 1, and evaluating to 1 is preserved by modus ponens, consequently every theorem evaluates to 1. Then we show that some 'truth' values give Pierce's law a value that is not 1, so it cannot be provable. There is one subtle part that we'll get to shortly. First, here is a 'truth table' for axiom scheme 1, *assuming P and Q are atomic.*

P	Q	$Q \supset P$	$P \supset (Q \supset P)$
0	0	1	1
0	a	0	1
0	b	0	1
0	1	0	1
a	0	1	1
a	a	1	1
a	b	a	1
a	1	a	t
b	0	1	1
b	a	1	1
b	b	1	1
b	1	b	1
1	0	1	1
1	a	1	1
1	b	1	1
1	1	1	1

The subtle point we mentioned comes up now. We have verified that axiom 1 instances evaluate to 1 only when the instances involve atomic formulas. What if we have something more complex? Think about a full argument.

As for axiom scheme 2, a verification table has 64 lines. Trust me, it works out. So, all axioms evaluate to 1.

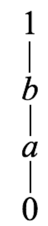
Next we show that evaluating to 1 is preserved by modus ponens. Suppose we are evaluating using some 'truth' assignment, and it makes X and $X \supset Y$ evaluate to 1. Since $X \supset Y$ evaluates to 1, we must be in one of the cases in the upper triangle of the $\supset$ table, where all entries are 1. Since the antecedent, X, evaluates to 1, we must be on the bottom row. Only one entry in the upper triangle of 1's is on the bottom row, and it is in the last column, where the consequent, Y, is 1. So, if X and $X \supset Y$ evaluate to 1 under one of our 'truth' assignments, so does Y.

It follows that, under any 'truth' assignment, every line of a proof must evaluate to 1. *Hence every theorem must evaluate to 1 under every 'truth' assignment.*

I'll leave it to you to check that if we assign P the value b and Q the value 1, Pierce's law, $((P \supset Q) \supset P) \supset P$ evaluates to b, and not to 1. Hence it is not provable.

Question To Think About The usual two valued truth tables characterize classical logic, and match up exactly with many axiomatizations. Do the two axioms we gave match the four valued truth table we gave?

The truth values we have using come from the following partial order, which (hopefully) we'll discuss in a later class. For now, don't read this.



2 Exercises

Consider the following propositional axiom system.

Axiom Schemes

1. $(P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R))$
2. $P \supset (Q \supset P)$
3. $P \supset (P \vee Q)$
4. $Q \supset (P \vee Q)$
5. $(P \supset R) \supset ((Q \supset R) \supset ((P \vee Q) \supset R)$

Rule of Inference $\dfrac{X \quad X \supset Y}{Y}$

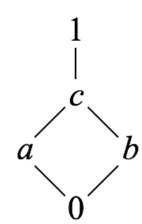
The problem is to show that $(P \supset Q) \vee (Q \supset P)$ is *not* provable. We use the following matrices.

$\supset$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	c	1
b	a	a	1	c	1
c	0	a	b	1	1
1	0	a	b	c	1

$\vee$	0	a	b	c	1
0	0	a	b	c	1
a	a	a	c	c	1
b	b	c	b	c	1
c	c	c	c	c	1
1	1	1	1	1	1

Designated value is only 1.

These matrices derive from the following partial order.



Show each at least one of the axiom schemes evaluates to value 1 under all assignments, and this is preserved under *modus ponens*. Find values for which $(P \supset Q) \vee (Q \supset P)$ is not provable.