Comments on Exercise Set 10

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May 3, 2018

Exercise set 10 had to do with first-degree entailment. Several people had their answers to question 3 marked down, and asked why. My explanation in class could have been better, and this is a more detailed discussion. The problem was to prove or disprove the statement below. More specifically, you were asked to either give a proof or a counter-example.

> Let A and B be formulas. If $\{A\} \vdash_{\mathsf{FDE}} B$ and $\{\neg B\} \vdash_{\mathsf{FDE}} \neg A$ then any valuation mapping A to *false* maps B to *false*.

Several people had answers that were along the following lines. Suppose $\{A\} \vdash_{\mathsf{FDE}} B$ and $\{\neg B\} \vdash_{\mathsf{FDE}} \neg A$ is true; we must show that A mapping to *false* under a valuation implies that B maps to *false* need not be the case. Well, consider a valuation v such that $v(A) = \textit{false}$ and $v(B) = \textit{true}$. That valuation is consistent with $\{A\} \vdash_{\mathsf{FDE}} B$ and $\{\neg B\} \vdash_{\mathsf{FDE}} \neg A$. It maps A to *false*, but does not map B to *false*. So it is possible that $\{A\} \vdash_{\mathsf{FDE}} B$ and $\{\neg B\} \vdash_{\mathsf{FDE}} \neg A$ is true but A mapping to *false* implies B mapping to *false* is not.

What is wrong with this as an answer? It is correct that such a valuation would take care of the problem. But, how do you know there is one? Remember, A and B can be *any* formulas. Suppose, for instance, that both A and B are atomic, and both are P. Trivially $\{A\} \vdash_{\mathsf{FDE}} B$ and $\{\neg B\} \vdash_{\mathsf{FDE}} \neg A$, but there is no valuation that would map A to *true* and B to *false*. Of course for this example it is the case that any valuation mapping A to *false* maps B to *false*, so this will not serve as a counter-example.

Again, suppose A and B are both atomic, but $A = P$ and $B = Q$, where P and Q are different. Then a valuation mapping A to *true* and B to *false* is possible, but $\{A\} \vdash_{\mathsf{FDE}} B$ and $\{\neg B\} \vdash_{\mathsf{FDE}} \neg A$ fails, so this will not serve as a counter-example.

The problem with the criterion that we want to map A to *false* but B to *true* is that you need to establish it is possible, that is, there really are formulas A and B such that $\{A\} \vdash_{\mathsf{FDE}} B$ and $\{\neg B\} \vdash_{\mathsf{FDE}} \neg A$, but one can map A to *false* and B to *true*. An actual counter-example is needed.

In class I gave an explicit counter-example. In fact, there is a simpler one. Take A to be P and B to be $P \vee Q$, where P and Q are atomic. It is easy to check that $\{P\} \vdash_{\mathsf{FDE}} P \vee Q$ and $\{\neg(P \vee Q)\} \vdash_{\mathsf{FDE}} \neg P$. And the valuation v where $v(P) = \textit{false}$ and $v(Q) = \textit{true}$ will meet the desired condition of mapping $A = P$ to *false* and $B = P \vee Q$ to *true*.