Exercise Set 8

Melvin Fitting

March 27, 2018

1. This is a problem about modal logic that cannot use tableaus because there aren't any 'reasonable' ones for the logic being considered. You must work with the semantics directly. The modal logic KX is characterized by the class of all models in which the accessibility relation is *dense*. Let $\mathcal{G}$ be the set of possible worlds of a model and $\mathcal{R}$ be its accessibility relation. The model is *dense* if, for any $\Gamma_1, \Gamma_2 \in \mathcal{G}$, if $\Gamma_1 \mathcal{R} \Gamma_2$ then there is some $\Gamma_3 \in \mathcal{G}$ such that $\Gamma_1 \mathcal{R} \Gamma_3$ and $\Gamma_3 \mathcal{R} \Gamma_2$. Loosely speaking, between any two possible worlds there is another. Call a model meeting the denseness condition a KX model.

   (a) Show that $\Box\Box X \supset \Box X$ is valid in all KX models. Hint: suppose this were not true and derive a contradiction.

   (b) Show that $\Box X \supset X$ is not valid in some KX model.

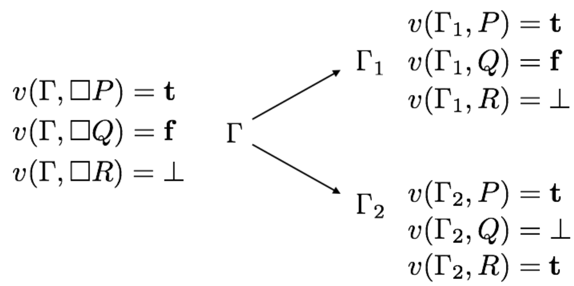
2. It is possible to combine many-valued logics with modal logics. This can be done in more than one way. Here is one of them. Let's call the models $\mathsf{K}K_3$ models., since we will be using frames with no accessibility conditions, and Kleene's strong three-valued logic. We will use $\perp$ instead of n for the third truth value—it represents a truth-value gap. We will not use $\Gamma \Vdash X$ and $\Gamma \nVdash X$ as notation, since it would be somewhat awkward to express that X has value $\perp$ at Γ. Instead we use the letter v to be a truth valuation and we write $v(\Gamma, X) = \mathbf{t}$ to indicate that X has the value t at Γ, and similarly with $\mathbf{f}$ and $\perp$.

   A $\mathsf{K}K_3$ model consists of a set $\mathcal{G}$ of possible worlds and an accessibility relation $\mathcal{R}$ with no special assumptions. But at each possible world Γ a truth value $v(\Gamma, P)$ is assigned to each propositional variable P using Kleene's strong three-valued logic: one of $\mathbf{t}$, $\perp$, or $\mathbf{f}$.

   At each possible world of a $\mathsf{K}K_3$ model truth for non-atomic formulas is evaluated using the K_3 truth tables, and the following.

$$v(\Gamma, \Box X) = \begin{cases} \mathbf{t} & \text{if } v(\Delta, X) = \mathbf{t} \text{ for every } \Delta \text{ such that } \Gamma\mathcal{R}\Delta \\ \mathbf{f} & \text{if } v(\Delta, X) = \mathbf{f} \text{ for some } \Delta \text{ such that } \Gamma\mathcal{R}\Delta \\ \perp & \text{otherwise} \end{cases}$$

   Here is an example. There are three possible worlds, Γ, Γ_1, and Γ_2. The accessibility relation is shown by arrows. At Γ_1 and Γ_2 truth values for the propositional variables P, Q, and R are given. Then truth values for $\Box P$, $\Box Q$, and $\Box R$ at Γ are calculated using the rules above.



Recall that in K_3, $\mathbf{t}$ is the designated truth value. For a set S of formulas and a single formula X let's say that X is a *consequence* of S in $\mathsf{K}K_3$ (written $S \Vdash_{\mathsf{K}K_3} X$) provided that for every $\mathsf{K}K_3$ model, every possible world Γ, and every valuation v, if $v(\Gamma, Y) = \mathbf{t}$ for every $Y \in S$ then $v(\Gamma, X) = \mathbf{t}$.

(a) In the example given above, what is the truth value of $\Box P$ at Γ_1, and why?

(b) Show that $\{\Box X, \Box Y\} \Vdash_{\mathsf{K}K_3} \Box(X \wedge Y)$.

(c) Suppose we define $\Diamond X$ to be an abbreviation for $\neg\Box\neg X$. Write out the truth conditions for $\Diamond X$ in $\mathsf{K}K_3$ moodels.