Proving Axiomatic Unprovability

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1 Quick Background

Before modern possible world semantics, algebraic semantics, and various other kinds of semantics were invented, what were called *matrices* provided a tool for proving things about logics. It was a tool that was awkward and unintuitive, but it influenced algebraic semantics, and it led directly to many-valued logics. Here we look at an example of matrix methods in use. Many-valued logics as such will be covered shortly.

2 Axiomatic Provability and Unprovability

Consider the following propositional axiom system.

Axiom Schemes

1. $P \supset (Q \supset P)$
2. $(P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R))$

Rule of Inference $\dfrac{X \quad X \supset Y}{Y}$

In this system $P \supset P$ is provable, but $((P \supset Q) \supset P) \supset P$ (Pierce's law) is not. How could we show this?

To show provability, we simply give a proof. Here is why we say $P \supset P$ is provable.

1. $(P \supset ((P \supset P) \supset P) \supset ((P \supset (P \supset P)) \supset (P \supset P))$ [$Q = (P \supset P)$, $R = P$ in scheme 2]
2. $P \supset ((P \supset P) \supset P)$ [$Q = P \supset P$ in scheme 1]
3. $(P \supset (P \supset P)) \supset (P \supset P)$ [modus ponens on 1, 2]
4. $P \supset (P \supset P)$ [$Q = P$ in scheme 2]
5. $P \supset P$ [modus ponens on 3, 4]

How do we show Pierce's law is *not* provable? Find some property every theorem has, but Pierce's law does not. Well, here's a non-standard truth table for implication, using four truth values. Two are the familiar truth, 1 and falsity, 0, and two are 'intermediate' values.

$\supset$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	b	1

Using this table, we show every theorem evaluates to 1 no matter which of the four 'truth values' we assign to propositional letters. We do this by showing that every axiom always evaluates to 1, and evaluating to 1 is preserved by modus ponens. Then we show that some 'truth values' give Pierce's law a value that is not 1, so it cannot be a theorem. First, here is a 'truth table' for axiom scheme 1, assuming P and Q are atomic.

P	Q	$Q \supset P$	$P \supset (Q \supset P)$
0	0	1	1
0	a	0	1
0	b	0	1
0	1	0	1
a	0	1	1
a	a	1	1
a	b	a	1
a	1	a	t
b	0	1	1
b	a	1	1
b	b	1	1
b	1	b	1
1	0	1	1
1	a	1	1
1	b	1	1
1	1	1	1

We have verified that axiom 1 instances evaluate to 1 only when the instances involve atomic formulas. What if we have something more complex? Think about a full argument.

As for axiom scheme 2, a verification table has 64 lines. Trust me, it works out. So, all axioms always evaluate to 1.

Next we show that evaluating to 1 is preserved by modus ponens. Suppose we are evaluating using some 'truth' assignment, and it makes X and $X \supset Y$ evaluate to 1. Since $X \supset Y$ evaluates to 1, we must be in one of the cases in the upper triangle of the $\supset$ table, where all entries are 1. Since the antecedent, X, evaluates to 1, we must be on the bottom row. Only one entry in the upper triangle of 1's is on the bottom row, and it is in the last column, where the consequent, Y, is 1. So, if X and $X \supset Y$ evaluate to 1 under one of our 'truth' assignments, so does Y.

It follows that, under any 'truth value' assignment, every line of a proof must evaluate to 1. *Hence every theorem must evaluate to 1 under every 'truth valuation'.*

I'll leave it to you to check that if we assign P the value b and Q the value a, Pierce's law, $((P \supset Q) \supset P) \supset P$ evaluates to b, and not to 1. Hence it is not provable.

Question To Think About The usual two valued truth tables characterize classical logic, and match up exactly with standard axiomatizations. Do the two axioms we gave match the four valued truth table we gave?

The truth values we have been using come from the following partial order, which (hopefully) we'll discuss in a later class.