First Degree Entailment, Continued

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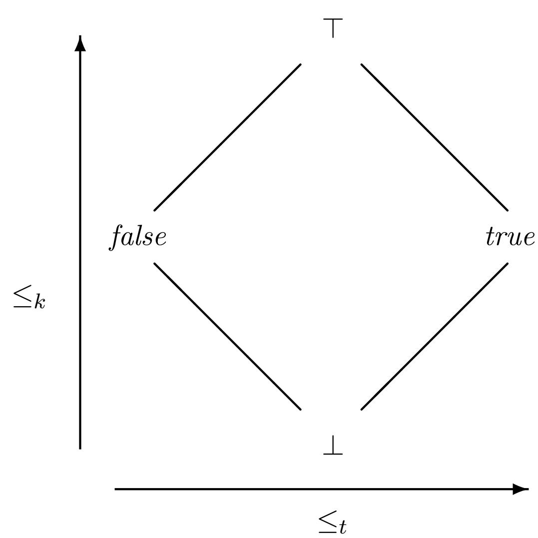
1 FDE Tableaus, the Ideas Behind Them

We have looked at truth tables for first degree entailment. Now we discuss one version of tableaus. There is a closely related system in the Graham Priest book, and we will say how it relates to ours later on. Recall, there are four truth values to FDE, and we arranged them in two orderings, truth-based and information-based. Here is the diagram from the end of the previous notes.



Only the logic operations concern us now. The information ordering is useful for certain applications, but for now it is an extra. We remind you that the truth values in the diagram can be thought of as sets of classical truth values.

$\bot$	for	$\emptyset$
false	for	$\{\mathbf{f}\}$
true	for	$\{\mathbf{t}\}$
$\top$	for	$\{\mathbf{f}, \mathbf{t}\}$

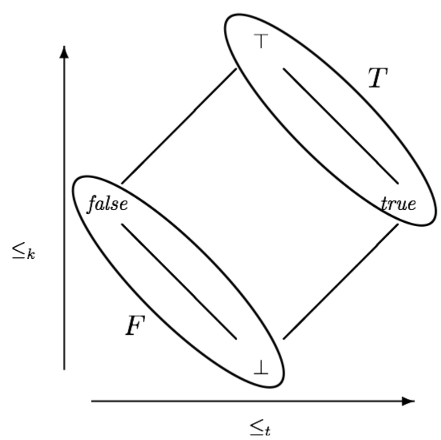
We also saw that, classically, conjunction and disjunction met the following conditions.

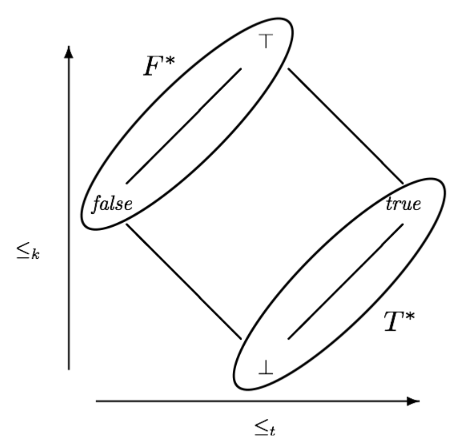
$$\begin{array}{lll} X \wedge Y \text{ has truth value } \mathbf{t} & \text{if and only if} & X \text{ has value } \mathbf{t} \text{ and } Y \text{ has value } \mathbf{t} \\ X \wedge Y \text{ has truth value } \mathbf{f} & \text{if and only if} & X \text{ has value } \mathbf{f} \text{ or } Y \text{ has value } \mathbf{f} \\ X \vee Y \text{ has truth value } \mathbf{t} & \text{if and only if} & X \text{ has value } \mathbf{t} \text{ or } Y \text{ has value } \mathbf{t} \\ X \vee Y \text{ has truth value } \mathbf{f} & \text{if and only if} & X \text{ has value } \mathbf{f} \text{ and } Y \text{ has value } \mathbf{f} \end{array} \tag{1}$$

These conditions completely determined the truth tables for conjunction and disjunction of the four truth values of FDE, as given in the earlier notes. But with four truth values, truth tables can be quite long, and tableaus will be a considerable time and space saver. One might expect that there will be four signs for our tableaus, and that is correct, but the signs do not correspond directly to the four values themselves.

Because of (1) it has been found to be natural to speak of "having **t**" as one case, and we will use the tableau sign T to correspond to it—thus T corresponds to *true* or $\top$. But then, should F correspond to "not having **t**" or to "having **f**"? They are not the same since the FDE truth values that do not include **t** are *false* and $\bot$, but the FDE truth values that do include **f** are *false* and $\top$. We will use F to correspond to the first of these alternatives, since then T and F will be disjoint, and we will have a natural tableau closure condition. Dually we'll use F^* for "having **f**", and T^* for "not having **f**". That is, F^* represents one of *false* or $\top$, while T^* represents one of *true* or $\bot$.

The sets of FDE truth values represented by our four signs are pictured in the following two diagrams.





2 Tableau Rules

Think of $T\,X$ informally as saying X has either $\top$ or *true* as its truth value. Either way, it says X has a truth value that contains **t**. Now by the first item of (1), $X \wedge Y$ will have a truth value containing **t** exactly when both X and Y do, so one of our tableau rules should be the following.

$$\frac{T\,X \wedge Y}{\begin{array}{c} T\,X \\ T\,Y \end{array}}$$

On the other hand, $T^*\,X$ informally says X does not have **f** as a truth value, so using the second item of (1) and a little reasoning, $X \wedge Y$ will not have **f** as a truth value exactly when X does not have **f** and Y does not have **f**. Thus we should also have the following similar looking rule (whose informal interpretation is not the same however).

$$\frac{T^*\,X \wedge Y}{\begin{array}{c} T^*\,X \\ T^*\,Y \end{array}}$$

In this way we can work out a full set of rules for conjunction and disjunction. Here they are.

$$\frac{T\,X \wedge Y}{\begin{array}{c}T\,X\\ T\,Y\end{array}} \qquad \frac{T^*\,X \wedge Y}{\begin{array}{c}T^*\,X\\ T^*\,Y\end{array}} \qquad \frac{F\,X \wedge Y}{F\,X \mid F\,Y} \qquad \frac{F^*\,X \wedge Y}{F^*\,X \mid F^*\,Y}$$

$$\frac{T\,X \vee Y}{T\,X \mid T\,Y} \qquad \frac{T^*\,X \vee Y}{T^*\,X \mid T^*\,Y} \qquad \frac{F\,X \vee Y}{\begin{array}{c}F\,X\\ F\,Y\end{array}} \qquad \frac{F^*\,X \vee Y}{\begin{array}{c}F^*\,X\\ F^*\,Y\end{array}}$$

Since the rules for starred and unstarred version look the same, one might wonder why we need two sets. The reason involves negation. Suppose we have $T\,\neg X$. We understand this as saying $\neg X$ has a truth value containing **t**, and thus is either $\top$ or *true*. Since $\neg\top = \top$ and $\neg \mathit{false} = \mathit{true}$, it must be that X itself has either $\top$ or *false* as its value, and thus is in the set we associated with F^*. So from $T\,\neg X$ we should infer $F^*\,X$ and not $F\,X$ as we would classically. Reasoning in this way we come up with the following set of negation rules.

$$\frac{T\,\neg X}{F^*\,X} \qquad \frac{T^*\,\neg X}{F\,X} \qquad \frac{F\,\neg X}{T^*\,X} \qquad \frac{F^*\,\neg X}{T\,X}$$

We can also produce rules for $\supset$, by understanding $X \supset Y$ as either $\neg X \vee Y$, or $\neg(X \wedge \neg Y)$, which behave the same in FDE. The rules are as follows.

$$\frac{T\,X \supset Y}{F^*\,X \mid T\,Y} \qquad \frac{T^*\,X \supset Y}{F\,X \mid T^*\,Y} \qquad \frac{F\,X \supset Y}{\begin{array}{c}T^*\,X\\ F\,Y\end{array}} \qquad \frac{F^*\,X \supset Y}{\begin{array}{c}T\,X\\ F^*\,Y\end{array}}$$

A tableau branch is closed if it contains both $T\,X$ and $F\,X$, or if it contains $T^*\,X$ and $F^*\,X$. (Think about what these mean intuitively.)

There are no valid formulas in FDE, but we did define consequence. As a reminder, for a set S of formulas and a single formula X, $S \vdash_{FDE} X$ provided every valuation that gives the members of S a designated truth value also gives X a designated truth value. To bring in tableaus, $S \vdash_{FDE} X$ if and only if there is a closed tableau beginning with $T\,Z$ for each $Z \in S$, and $F\,X$.

3 Examples

We look at two examples from our earlier truth table notes.

Example 3.1 $\{\neg(P \vee Q), \neg R\} \vdash_{FDE} \neg(P \vee R)$ holds. Here is a closed tableau. Please check the steps.

$$\begin{array}{c}
T\,\neg(P \vee Q)\\
T\,\neg R\\
F\,\neg(P \vee R)\\
F^* P \vee Q\\
F^* R\\
T^* P \vee R\\
F^* P\\
F^* Q\\
\swarrow \quad \searrow\\
T^* P \qquad\qquad T^* R
\end{array}$$

Example 3.2 $\{P, \neg P \vee Q\} \vdash_{FDE} Q$ does not hold. In the earlier notes we got counter-examples from a truth table. We now get them from a tableau construction.

$$T\,P$$
$$T\,\neg P \vee Q$$
$$F\,Q$$

$T\,\neg P$	$T\,Q$
$F^*\,P$	

The right branch is closed, but the left one is not. We get counter-examples by giving propositional letters values according to what the left branch says. It has $F\,Q$, so Q can be either *false* or $\bot$, the values we associated with F. It also has both $T\,P$ and $F^*\,P$. From the first of these, P should be one of $\top$ or *true*. From the second it should be one of *false* or $\top$. Hence P should be exactly $\top$. We thus have two counter-examples, P is $\top$ and Q is *false*, or P is $\top$ and Q is $\bot$. You should check that these are counter-examples.

4 Connections With the Graham Priest System

The tableau system in the book is very similar to the one described above. The differences are really those of notation. Of course the system in the book does not use signs in front at all. But where we write $T\,X$ or $F\,X$, the book will use $X, +$ and $X, -$. This is a trivial difference. More importantly, we have used four signs, T, F, T^*, and F^*. The book FDE tableau system also has the same four-fold division behind it, but instead of four distinct symbols, the negation symbol is overloaded. If you replace $F^*\,X$ by $T\,\neg X$, and $T^*\,X$ with $F\,\neg X$, our tableaus become book style (except for the use of $+/-$ as just noted.)

For instance, consider Example 3.1 with the translation just described. It becomes the following.

$$T\,\neg(P \vee Q)$$
$$T\,\neg R$$
$$F\,\neg(P \vee R)$$
$$T\,\neg(P \vee Q)$$
$$T\,\neg R$$
$$F\,\neg(P \vee R)$$
$$T\,\neg P$$
$$T\,\neg Q$$

$F\,\neg P$	$F\,\neg R$

This is a literal translation. There are redundancies, but eliminating them, and introducing some applications of deMorgan laws, produces a proper proof in the system of the book. We have used four signs out of a feeling that it is pedagogically clearer. There is nothing deep about the differences between the systems.