Notes on Regular Logics

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When Kripke's modal semantics was first introduced there were well-known modal logics that it could not handle. In [4] an extension of his semantics was presented, and it allowed for the modeling of an important group of these. The key idea was to introduce *non-normal worlds*. Kripke's version will be discussed here, and is in Chapter 4 of Graham Priest's book. Since then, other kinds of non-normal worlds have been created—it is quite a useful tool. We will only discuss the historically first version.

C. I. Lewis had introduced a series of five modal logics, called $\mathsf{S1} - \mathsf{S5}$. Kripke's first semantics, [2], could model $\mathsf{S5}$, and his full version, [3], could handle $\mathsf{S4}$ as well as many other important modal logics. Earlier Lemmon, in [5], introduced some new and interesting modal logics and related some of them to the Lewis systems $\mathsf{S2}$ and $\mathsf{S3}$. The basic Lemmon logic is today called C (he called it $\mathsf{C2}$, and Priest calls it N). Two natural extensions are CT and $\mathsf{CT4}$, and we will see all these below. Kripke provided a direct semantics for these and indirectly, using Lemmon's work, for the Lewis systems $\mathsf{S2}$ and $\mathsf{S3}$. ($\mathsf{S1}$ did eventually acquire a semantics, but it is not particularly natural, and the logic has faded in interest).

1 Regular Logics

I know we are not working with axiom systems in this class, but sometimes they do provide insight. *Normal* modal logics are those resulting by adding axiom schemes to K, where K is axiomatized using classical propositional logic, the axiom scheme $\Box(X \supset Y) \supset (\Box X \supset \Box Y)$, the rule of *modus ponens*, and the rule of *necessitation*: from X to conclude $\Box X$. (Here we understand $\Diamond$ as being defined from $\Box$.) In class we showed that *regularity* is a defined rule: from $X \supset Y$ conclude $\Box X \supset \Box Y$.

Lemmon's new logics are *regular* logics. These are axiomatized by adding schemas to C, where C is just like K except that the rule of regularity replaces the rule of necessitation. It is possible to show that no regular logic has any theorem of the form $\Box X$. In particular, $\Box\top$ is not provable, where $\top$ is a proposition that is always true. Then in regular logics nothing is simply necessary, but things are necessary relative to the necessity of something else. For instance, $\Box(X \wedge Y) \supset \Box X$ is a theorem of C because it follows from the tautology $(X \wedge Y) \supset X$ using the rule of regularity. It says that X is necessary relative to the necessity of $X \wedge Y$, a kind of simplification. (Think about what this says when $Y = X$.)

Since tautologies are true at every world of any standard Kripke model, $\Box X$ will be true at every world of every Kripke model whenever X is a tautology. Yet it is not provable in a regular logic. So the usual Kripke semantics is inadequate to handle regular logics.

2 Regular Semantics

The semantics of Kripke that we have been using is called *normal* semantics, when we need a name for it. Now we make an addition to the machinery to get the *regular* semantics. In a model, a subset of the entire set of possible worlds is identified as the *normal* worlds—the subset is called N.[1] There may be possible worlds not in N—these are called *non-normal* worlds. The rules of valuation for formulas at possible worlds are modified for modal operators (propositional connectives still behave truth functionally, as before). At normal worlds, things are as they were. At non-normal worlds *everything is possible, nothing is necessary.* That is, if Γ is non-normal, $\Gamma \Vdash \Diamond X$ and $\Gamma \not\Vdash \Box X$, for every X. Note that if there are no non-normal worlds, we just have the usual Kripke semantics.

The axiom $\Box(X \supset Y) \supset (\Box X \supset \Box Y)$ is true at every world of a regular model (even if there are non-normal worlds). The argument that it is true at a normal world is exactly as it was before. But also, if Γ is a non-normal world it must be true there because otherwise we would have $\Gamma \Vdash \Box(X \supset Y)$ but $\Gamma \not\Vdash \Box X \supset \Box Y$, but we can't have a necessitated formula true at a non-normal world. By a similar argument, validity in regular models is preserved by the rule of regularity.

The most general regular semantics, with no special conditions on the accessibility relation, is what corresponds to the Lemmon logic C.

3 A Tableau System for C

The following destructive tableau system was (I believe) introduced in [1]. Call a tableau branch *normal* if it contains a signed formula of one of the forms $T\,\Box X$ or $F\,\Diamond X$. (Priest calls the analogous condition on his type of tableaus *inhabited.*) In a C tableau the following rules using the definition of $S^\sharp$ for K *can only be used on a normal branch.*

$$\frac{S, F\,\Box X}{S^\sharp, F\,X} \qquad \frac{S, T\,\Diamond X}{S^\sharp, T\,X}$$

For example, here is a proof in C of $\Box(\Box X \wedge \Box Y) \supset \Box\Box(X \wedge Y)$.

$$\begin{array}{c}
F\,\Box(\Box X \wedge \Box Y) \supset \Box\Box(X \wedge Y) \\
T\,\Box(\Box X \wedge \Box Y) \\
F\,\Box\Box(X \wedge Y) \\
\hline
T\,\Box X \wedge \Box Y \\
F\,\Box(X \wedge Y) \\
T\,\Box X \\
T\,\Box Y \\
\hline
F\,X \wedge Y \\
T\,X \\
T\,Y \\
\swarrow \quad \searrow \\
F\,X \qquad\quad F\,Y
\end{array}$$

Notice that every time a destructive rule is applied above, the uncancelled formulas on the branch make the branch nornal.

[1] Historically instead of singling out the normal worlds, the non-normal worlds were specified instead, and were called *queer worlds*. This terminology is no longer used.

The normal tableau system for T adds to the rules for K the following.

$$\frac{T\,\Box X}{T\,X} \qquad \frac{F\,\Diamond X}{F\,X}$$

For CT we add the same rules to those for C.

For K4 the definition of $S^\sharp$ for K was changed to $S^\sharp = \{T\,X, T\,\Box X \mid T\,\Box X \in S\} \cup \{F\,X, F\,\Diamond X \mid F\,\Diamond X \in S\}$. That is, we keep the original signed formula as well as its reduction. For the regular analog, C4, this gets modified to the following.

$$S^\sharp = \{T\,X, T\,\Box\top \supset \Box X\} \cup \{F\,X, F\,\Box\top \wedge \Diamond X \mid F\,\Diamond X \in S\}$$

Then we use the rules for C but with this definition of $S^\sharp$. Please Note: we have not made use of $\top$ in tableaus before. We need one more closure rule for it: a branch closes if it contains $F\,\top$.

Finally, among the regular logics we treat, the system CT4 is the analog of the normal system S4. Just as S4 combined the changes for T and K4, here too CT4 combines the changes of CT and C4.

Here is a closed CT4 tableau for $\Box(P \supset Q) \supset \Box(\Box P \supset \Box Q)$.

$$\begin{array}{c}
F\,\Box(P \supset Q) \supset \Box(\Box P \supset \Box Q) \\
T\,\Box(P \supset Q) \\
F\,\Box(\Box P \supset \Box Q) \\
\hline
T\,P \supset Q \\
T\,\Box\top \supset \Box(P \supset Q) \\
F\,\Box P \supset \Box Q \\
T\,\Box P \\
F\,\Box Q
\end{array}$$

$$\begin{array}{cc}
F\,\Box\top & T\,\Box(P \supset Q) \\
\hline
T\,P & T\,P \\
T\,\Box\top \supset \Box P & T\,\Box\top \supset \Box P \\
F\,\top & T\,P \supset Q \\
 & T\,\Box\top \supset \Box(P \supset Q) \\
 & F\,Q
\end{array}$$

$$\begin{array}{cc}
F\,P & T\,Q
\end{array}$$

4 S2 and S3

Lemmon showed that X is a theorem of the Lewis system S2 if and only if $\Box\top \supset X$ is a theorem of CT, and it is a theorem of S3 if and only if $\Box\top \supset X$ is a theorem of CT4. This means we have tableau systems for S2 and S3: just use those for CT and CT4, but to prove X construct a closed tableau for $\Box\top \supset X$. And we have a semantics too. It is easy to see that in any regular model, $\Box\top$ will be true at exactly the normal possible worlds. Then, $\Box\top \supset X$ will be valid in a given regular model just in case X is true at all the normal worlds of it. So our semantics for S2 is to use the models for CT, but only consider normal worlds. Similarly for S3, using models for CT4. What, then, is the role of the non-normal worlds? Well, evaluating truth at a normal world might force us to look at non-normal ones.

References

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