$$\frac{|S, \gamma(c)|}{|S, \gamma|}$$

Where c is any constant symbol.

$$\frac{|S, \delta(c)|}{|S, \delta|}$$

Where c is a constant symbol that does not occur in $\{S, \delta\}$.

As we said, the quantifier rules are deceptively simple. Since there are (we assume) infinitely many constant symbols available, if the γ-rule can be applied at all, it can be applied in infinitely many different ways. This means a tableau can never be completed in a finite number of steps. Essentially, this is the source of the undecidability of classical first-order logic.

3 Modern History

After reaching a stable form in the classical case, the next stage in the development of tableaus was the extension to various non-classical logics. In this section we sketch a few such systems, say how they came about, and present the intuitions behind them. The particular systems chosen illustrate the variety of extra machinery that has been developed for and added to tableau systems: reinterpreting signs, generalizing signs, modifying closure rules, allowing trees to change in ways other than simple growth, adjoining 'side' information, and using pairs of coupled trees.

3.1 Intuitionistic Logic

Sequent calculi for intuitionistic logic were around from the beginning—Gentzen and Beth both developed them—so it is not surprising that a tableau version would be forthcoming (see Chapter **??**). The first explicitly presented as such seems to be in the 1969 book of Fitting [24]. In this the signed tableau system of Smullyan was adapted, with the signs given a new informal interpretation. In the resulting tableau system, proof trees were allowed to shrink as well as grow.

For both Lis and Smullyan, signs primarily were a device to keep track of left and right sides of sequents, without explicitly using sequent notation. Signs also had an intuitive interpretation that was satisfying: $T\,X$ and $F\,X$ can be thought of as asserting that X is true or false in a model. But now, think of $T\,X$ as informally meaning that X is *intuitionistically* true, that is, X has been given a proof that an intuitionist would accept. Likewise think of $F\,X$ as asserting the opposite: X has not been given an intuitionistically acceptable proof. (This is quite different from assuming X is intuitionistically refutable, by the way.) Some tableau rules are immediately suggested. For instance, intuitionists read disjunction constructively: to prove $X \vee Y$ one should either prove X or prove Y (see [39]). Then if we have $T\,X \vee Y$ in a tableau, informally $X \vee Y$ has been intuitionistically proved, hence this is the case for one of X or Y, so the tableau branch splits to $T\,X$ and $T\,Y$. Likewise if we have $F\,X \vee Y$, we do not have an intuitionistically acceptable proof of $X \vee Y$, so we can have neither a proof of X nor of Y, and so we can add both $F\,X$ and $F\,Y$ to the branch. That is, intuitionistic rules for disjunction look like classical ones! The same is the case for conjunction. But things begin to get interesting with implication. We quote Heyting [39].

> "The *implication* $\mathfrak{p} \rightarrow \mathfrak{q}$ can be asserted, if and only if we possess a construction $\mathfrak{r}$, which, joined to any construction proving $\mathfrak{p}$ (supposing that the latter be effected), would automatically effect a construction proving $\mathfrak{q}$. In other words, a proof of $\mathfrak{p}$, together with $\mathfrak{r}$, would form a proof of $\mathfrak{q}$."

If $T\,X \supset Y$ occurs in a tableau, informally we have a proof of $X \supset Y$, and so we have a way of converting proofs of X into proofs of Y. Then, in our present state of knowledge, either we are not able to prove X, or we are, in which case we can provide a proof of Y as well. That is, the tableau branch splits to $F\,X$ and $T\,Y$, just as it does classically.

Now suppose $F\,X \supset Y$ occurs in a tableau. Then intuitively, we do not have a mechanism for converting proofs of X into proofs of Y. This does not say anything at all about whether we are able to prove X. What it says is that someday, not necessarily now, we may discover a proof of X without being able to convert it to a proof of Y. That is, *someday* we could have both $T\,X$ and $F\,Y$. We are talking about a possible future state of our mathematical lives. Now, as we move into the future, what do we carry with us? If we *have not* proved some formula Z, this is not necessarily a permanent state of things—tomorrow we may discover a proof. But if we *have* proved Z, tomorrow this will still be so—a proof remains a proof. Thus, when passing from a state to a possible future state, signed formulas of the form $T\,Z$ should remain with us; signed formulas of the form $F\,Z$ need not. This suggests the following tableau rule: if a branch contains $F\,X \supset Y$, add both $T\,X$ and $F\,Y$, but first delete all signed formulas on the branch that have an F sign. (Negation has a similar analysis.)

We must be a little careful with this notion of formula deletion, though. The tree representation for tableaus that we have been using marks the presence of a node by using a formula as a label. If we simply delete formulas, information about node existence and tableau structure could be lost. What we do instead, when using this representation of tableaus, is leave deleted formulas in place, but check them off, placing a $\surd$ in front of them. (Of course, when using the set of sets representation for tableaus from section 1.2, things are simpler: just replace one set by another, since there is no structure sharing.) There is still one more problem, though. It may happen that a formula should be deleted on one branch, but not on another, and using the tree representation for tableaus, its presence on both branches might be embodied in a single formula occurrence. In this case, check it off where it occurs, and add a fresh, unchecked occurrence to the end of the branch on which it should not be deleted.

To state the intuitionistic rules formally we use notation from [24], if S is a set of signed formulas, let S_T be the set of T-signed members of S. We write $S, F\,Z$ to indicate a tableau branch containing the signed formula $F\,Z$, with S being the set of remaining formulas on the branch. Now, the propositional intuitionistic rules are these.

Conjunction $\dfrac{S, T\,X \wedge Y}{S, T\,X, T\,Y}$ $\dfrac{S, F\,X \wedge Y}{S, F\,X \mid S, F\,Y}$

Disjunction $\dfrac{S, T\,X \vee Y}{S, T\,X \mid S, T\,Y}$ $\dfrac{S, F\,X \vee Y}{S, F\,X, F\,Y}$

Implication $\dfrac{S, T\,X \supset Y}{S, F\,X \mid S, T\,Y}$ $\dfrac{S, F\,X \supset Y}{S_T, T\,X, F\,Y}$

Negation $\dfrac{S, T\,\neg X}{S, F\,X}$ $\dfrac{S, F\,\neg X}{S_T, T\,X}$

Then unlike with classical tableaus, as far as usable formulas are concerned, intuitionistic tableaus can shrink as well as grow. If a branch contains, say, both $F\,X \supset Y$ and $F\,A \supset B$, using an implication rule on one will destroy the other. It is possible to make a bad choice at this point and miss an available proof. For completeness sake, both possibilities must be explored. This is the analog of Beth's *disjunctive* branching, see section 2.2.2. As a matter of fact, the tableau rules above correspond to Beth's rules in the same way that the classical tableau rules of Lis and Smullyan correspond to Beth's classical rules.

A tableau branch is closed if it contains $T\,X$ and $F\,X$, for some formula X, *where neither is a deleted signed formula*. Of course the intuitive idea is somewhat different than in the classical case now: the contradiction is that an intuitionist has both verified and failed to verify X. Nonetheless, a contradiction is still a contradiction.

We conclude with two examples, a non-theorem and a theorem. The non-theorem is $(\neg X \supset X) \supset X$. A tableau proof attempt begins with $F\,(\neg X \supset X) \supset X$. An application of the $F \supset$ rule causes this very F-signed formula to be deleted, and produces 2 and 3 below. Use of $T \supset$ on 2 gives 4 and 5. The right branch is closed, but closure of the left branch is impossible since an application of $F\neg$ to 4 causes deletion of 3.

$\surd$ 1. $F\,(\neg X \supset X) \supset X$
2. $T\,\neg X \supset X$
3. $F\,X$

4. $F\,\neg X$ | 5. $T\,X$

The formula $(\neg X \supset X) \supset \neg\neg X$, on the other hand, is a theorem. A proof of it begins with $F\,(\neg X \supset X) \supset \neg\neg X$, then continues as follows.

$\surd$ 1. $F\,(\neg X \supset X) \supset \neg\neg X$
2. $T\,\neg X \supset X$
$\surd$ 3. $F\,\neg\neg X$
4. $T\,\neg X$

5. $F\,\neg X$ | 6. $T\,X$; 7. $F\,X$

An application of the $F \supset$ rule deletes 1 and adds 2 and 3. Then $F\neg$ applied to 3 deletes it and adds 4. The $T \supset$ rule applied to 2 adds 5 and 6, and finally, the $T\neg$ rule applied to 4 adds 7. The tableau is closed because of 4 and 5, and 6 and 7, none of which are checked off.

At the cost of a small increase in the number of signs, Miglioli, Moscato, and Ornaghi have created a tableau system for intuitionistic logic that is more efficient than the one presented above [56, 57, 58]. In addition to the signs T and F, one more sign, F_c, is introduced. In terms of Kripke models, we can think of $T\,X$ as true at a possible world if X is true there, in the sense customary with intuitionistic semantics. Likewise we can think of $F\,X$ as true at a possible world if X is *not* true there. But now, think of $F_c\,X$ as true at a world if $\neg X$ is true there. This requires additional tableau rules, but reduces duplications inherent in the tableau system without the additional sign. Without going into details, this should suggest some of the flexibility made possible by the use of signed formulas, a topic to be continued in the next section.

3.2 Many-Valued Logic

The signs of a signed tableau system can be reinterpreted, as in intuitionistic logic, and they can be extended, as happened in many-valued logic (see Chapter **??**). Finitely-valued Łukasiewicz logics were given a tableau treatment by Suchon in 1974 [87]. Surma considered a more general situation in 1977, [88], and this was further developed by Carnielli in 1987,