Final Exam—Non-Classical Logics

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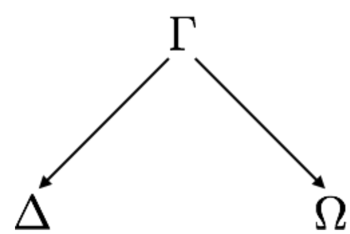
Question 1. (*14 points*) In classical logic some connectives are definable from others. For example, $X \vee Y$ is equivalent to $\neg X \supset Y$ and also to $\neg(\neg X \wedge \neg Y)$. In intuitionistic logic, all of $\wedge, \vee, \neg, \supset$ are independent. This exercise establishes that $\vee$ is independent of the others.

We use the following frame for an intuitionistic model, with three possible worlds. Each world is accessible from itself (not explicitly shown in the diagram) and the arrows show accessibility otherwise.



To make this into a model we need to supply a valuation saying what atomic formulas are true at what worlds. Suppose we have done this, and we now have a model. Let us call a formula Z *bad* if Z is true at exactly the worlds Δ and Ω. That is, $\Delta \Vdash Z$, $\Omega \Vdash Z$, but $\Gamma \nVdash Z$. Call Z *good* if it is not bad. You are to show the following.

(a) $\neg X$ is never bad.

(b) If $X \supset Y$ is bad, so is Y.

(c) If $X \wedge Y$ is bad, so is one of X or Y.

(d) If X and Y are good, so are $X \supset Y$, $\neg X$, and $X \wedge Y$.

With your questions done, the impossibility of defining $\vee$ goes as follows. Let P and Q be atomic, and suppose we had a formula defining $P \vee Q$. That is, we have a formula $\Phi(P, Q)$ built up from just P and Q using $\wedge$, $\neg$ and $\supset$ so that $(P \vee Q) \equiv \Phi(P, Q)$ is an intuitiuonistic logic theorem. Now build a model on the frame pictured above by setting $\Delta \Vdash P$, $\Omega \Vdash Q$, and no other cases of atomic truth hold. In the resulting model, P is not bad since it only holds at one of Δ and Ω, so P is good. Similarly Q is good. Then by *Question 1*.d above, it follows that $\Phi(P, Q)$ must be good. But $P \vee Q$ is true at exactly Δ and Ω, so $P \vee Q$ is bad. Then $P \vee Q$ and $\Phi(P, Q)$ cannot be equivalent since they do not hold at the same set of worlds.

Question 2. (*10 points*) The following is valid in $\mathsf{S5}$.

$$\Diamond(P \wedge \Diamond(Q \wedge \Box R)) \supset (\Diamond(P \wedge R) \wedge \Diamond(Q \wedge R))$$

(a) Prove this by reasoning directly about possible world models.

(b) Also prove it using tableaus.

Question 3. (*10 points*) The modal logic S4.3 is characterized by frames that are reflexive, transitive, and linear, where linearity means: for all $\Gamma, \Delta, \Omega \in \mathcal{G}$, if $\Gamma\mathcal{R}\Delta$ and $\Gamma\mathcal{R}\Omega$ either $\Delta\mathcal{R}\Omega$ or $\Omega\mathcal{R}\Delta$. Show that

$$\Diamond\Box(P \supset Q) \supset (\Diamond\Box P \supset \Diamond\Box Q)$$

is valid in all S4.3 frames. (Note: we gave no tableau system for this logic. You must use models directly.)

Question 4. (*12 points*) Let S be a set of formulas and X be a single formula, with none of the formulas involving implication or modal operators. Show that if $S \Vdash_{\mathsf{FDE}} X$ then some propositional letter in X must also occur in a formula in S. Hint: use the four-valued semantics with $\{\mathit{true}, \top\}$ as designated values. Don't use the tableau system.

Question 5. (*12 points*) Let S be a set of formulas and X be a single formula, with none of the formulas involving implication or modal operators. Prove or disprove that if $S \Vdash_{\mathsf{K}_3} X$ then some propositional letter in X must also occur in a formula in S. K_3 is the strong Kleene three-valued logic.

The rest of the questions concern justification logic. Assume the logic is LP, formulated by taking *all* tautologies as axioms. Assume there is a constant specification that assigns a constant to every axiom of this logic.

Question 6. (*14 points*) Show the following strong form of the internalization theorem: if X is a theorem of LP then there is a justification term t such that $t{:}X$ is a theorem, where t is built up from justification constants using the operation symbols $\cdot$ and $!$, and does not contain justification variables or $+$.

Question 7. (*14 points*) $\Box(P \supset Q) \supset (\Box\Box P \supset \Box\Box Q)$ is a theorem of S4. Your problem is to find a normal realization for it, provable in LP. Here is an informal argument for the modal theorem.

(a) $\Box(P \supset Q) \supset (\Box P \supset \Box Q)$ is a theorem. That $\Box$ distributes over $\supset$ is usually taken as an axiom in normal modal logics.

(b) $\Box\Box P \supset \Box P$ is a theorem. More generally, $\Box X \supset X$ is a theorem of S4.

(c) $\Box Q \supset \Box\Box Q$ is a theorem. It is a special case of $\Box X \supset \Box\Box X$, which is usually taken as an axiom for S4.

(d) Now $\Box(P \supset Q) \supset (\Box\Box P \supset \Box\Box Q)$ follows from *Question 7*.a, *Question 7*.b, and *Question 7*.c using classical logic.

Now find realizations for *Question 7*.a, *Question 7*.b, and *Question 7*.c in LP, then combine them to get a realization for *Question 7*.d.

Question 8. (*14 points*) Here is an example of what is sometimes called a semi-replacement result for S4. Consider the formula $\Box(\Box P \vee Q)$. (It is not a theorem.) Also consider the formula $X \supset Y$, where we assume X and Y are such that this is a theorem of S4. We can replace propositional letter P by either X or Y, and doing so preserves the implication in the following sense: $\Box(\Box X \vee Q) \supset \Box(\Box Y \vee Q)$ is a theorem of S4. Here is a sketch of the verification of this.

(a) $X \supset Y$, assumed to be a theorem of $\mathsf{S4}$.

(b) $\Box(X \supset Y)$ is a theorem, using the necessitation rule on *Question 8*.a. Then $\Box X \supset \Box Y$ follows using the distributivity of $\Box$ over $\supset$.

(c) $(\Box X \vee Q) \supset (\Box Y \vee Q)$ follows from *Question 8*.b using classical logic.

(d) $\Box[(\Box X \vee Q) \supset (\Box Y \vee Q)]$ follows from *Question 8*.c using the necessitation rule. Then $\Box(\Box X \vee Q) \supset \Box(\Box Y \vee Q)$ follows using the distributivity of $\Box$ over $\supset$.

Your problem is to see what all this turns into in LP. Consider the LP formula $x{:}(y{:}P \vee Q)$. Assume $X \supset Y$ is a theorem of LP. Show there are justification terms t and u so that $x{:}(y{:}X \vee Q) \supset t{:}(u{:}Y \vee Q)$ is a theorem of LP.